Carleton University Physics Department

PHYS 4708 (Winter 2016, H. Logan)

Homework assignment #5

Handed out Wed. Mar. 2; due Wed. Mar. 16, 2016 at the start of class.
Problems are worth 5 points each unless noted otherwise.

1. *[10 points]* (Gasiorowicz 3rd ed. problem 15-3) Consider a particle in an infinite well in 1 dimension, with $V(x) = 0$ for $0 \leq x \leq a$ and $V(x) = \infty$ everywhere else. A tilt of the potential in the range $0 \leq x \leq a$ is turned on and then off according to

$$V_1(x,t) = \lambda \left(x - \frac{a}{2} \right) e^{-t^2/\tau^2}. \tag{1}$$

   (a) Calculate the probability that a particle initially in the ground state ($n = 1$) ends up in the first excited state ($n = 2$).

   (b) What is the probability that a particle initially in the ground state ends up in the second excited state ($n = 3$)?

   (c) What happens to these results as $\tau \to \infty$? *(This shows that for a very slowly varying perturbation, transitions become strongly suppressed. The next problem gives a more general examination of this.)*

2. (Very similar to Gasiorowicz 3rd edition problem 15-6) This problem illustrates the *adiabatic theorem*. The theorem states that if the Hamiltonian is changed very slowly from H_0 to H, then a system in a given eigenstate of H_0 goes over into the corresponding eigenstate of H, but does not make any transitions. To be specific, consider the ground state, so that

$$H_0 \phi_0 = E_0 \phi_0. \tag{2}$$

   Let $V(t) = V f(t)$, where $f(t)$ is a slowly varying function that interpolates monotonically from $f = 0$ at $t = 0$ to $f = 1$ as $t \to \infty$ as shown in the graph at the bottom of page 244 of the textbook. If the ground state of $H \equiv H_0 + V$ is $|w_0\rangle$, the theorem states that

$$|\langle w_0 | \psi(t) \rangle| \to 1 \tag{3}$$

   as the time variation of f becomes infinitely slow.

   (a) Show that

$$\frac{1}{i\hbar} \int_0^t dt' e^{i(E_m^0 - E_0^0)t'/\hbar} f(t') \to \frac{-e^{i(E_m^0 - E_0^0)t/\hbar}}{E_m^0 - E_0^0} \tag{4}$$

   for times t such that $f(t) = 1$. Use the fact that

$$\frac{df(t')}{dt'} \ll \frac{E_m^0 - E_0^0}{\hbar} f(t'). \tag{5}$$

   Note: it's most straightforward to use integration by parts; that is, to write

$$e^{i\omega t'} = \frac{1}{i\omega} \frac{d}{dt'} e^{i\omega t'}. \tag{6}$$

(b) Calculate $\psi(t)$ using time-dependent perturbation theory, equations (15-3) and (15-10) in the text. Compare this with the formula for the ground state of H as given by time-independent perturbation theory in equation (11-15), which here reads

$$|w_0\rangle = |\phi_0\rangle + \sum_{m\neq 0} \frac{\langle\phi_m|V|\phi_0\rangle}{E_0^{(0)} - E_m^{(0)}}|\phi_m\rangle. \tag{7}$$

Thus show that, in the adiabatic limit,

$$|\psi(t)\rangle \to |w_0\rangle e^{-iE_0^{(0)}t/\hbar}. \tag{8}$$

3. *[10 points]* (similar to Gasiorowicz 3rd ed. problem 16-1) Consider a particle of mass m in a three-dimensional harmonic oscillator with potential energy $m\omega^2 r^2/2$. Because the potential is radial, the energy eigenstates can be written in terms of spherical harmonics,

$$\psi(\vec{r}) = R_{n_r\ell}(r)Y_{\ell m_\ell}(\theta, \phi), \tag{9}$$

where $R_{n_r\ell}(r)$ is the radial wavefunction (not the same as the radial wavefunctions of the hydrogen atom). The energy eigenvalues are $E = \hbar\omega(2n_r + \ell + 3/2)$, where $n_r = 0, 1, 2, \ldots$ and $\ell = 0, 1, 2, \ldots$.

(a) Work out the combinations of n_r and ℓ that contribute to the first five energy levels.

(b) Now suppose the particle has charge q and the harmonic oscillator is placed in a *weak* constant uniform magnetic field $\vec{B}$. Write down the Schrödinger equation in the presence of this field and treat the new term(s) as a time-independent perturbing Hamiltonian H_1. Compute the first-order shifts $E^{(1)}$ in the energies due to this perturbation, and sketch the resulting spectrum (assume that the magnetic energy shifts are small compared to the harmonic oscillator level spacings).

Hint: For a constant uniform magnetic field you can use $\vec{A} = -\vec{r} \times \vec{B}/2$. For a weak field, you can neglect terms of second order in $\vec{A}$. The vector identity $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$ may come in handy.