Carleton University Physics Department
PHYS 4708 (Winter 2016, H. Logan)
Homework assignment #4

Handed out Wed. Feb. 10; due Wed. Mar. 2, 2016 at the start of class.
Problems are worth 5 points each unless noted otherwise.

1. Consider two identical non-interacting spin-3/2 particles.

   (a) Assume that the two particles are produced in an S-wave configuration, i.e., that their orbital angular momentum is zero. What are the values of the total spin $\vec{S}_{\rm tot} = \vec{S}_1 + \vec{S}_2$ that are allowed by the Pauli principle? (You can use the table of Clebsch-Gordan coefficients; see the course webpage for a link.)

   (b) Now assume that the two particles are produced in a P-wave configuration, i.e., that their orbital angular momentum is $\ell = 1$. What are the values of the total spin $\vec{S}_{\rm tot} = \vec{S}_1 + \vec{S}_2$ that are allowed by the Pauli principle? What are the allowed values of the total angular momentum (combining spin and orbital)?

2. *[10 points]* (similar to Gasiorowicz 3rd ed. problem 13-3) Consider two non-interacting electrons in a one-dimensional infinite potential well.

   (a) What is the ground-state wave function of the two-electron system if the two electrons are in the *same* spin state?

   (b) Re-express the ground-state wave function in terms of the relative coordinate $x \equiv x_1 - x_2$ and the centre-of-mass coordinate $X \equiv (x_1 + x_2)/2$. Square to obtain the joint probability density $P(x, X)$ and integrate over X to find the probability density for the relative coordinate $P(x)$. *(Hint: be very careful with the range of integration of X: it will help to sketch for yourself the region in x and X that is accessible by the system.)*

   (c) Make a sketch of $P(x)$ (you can plot it using a computer). What is the average separation between the two electrons? *(This last calculation is probably best done numerically. You can also check numerically that $P(x)$ is properly normalized.)*

3. (Similar to Gasiorowicz 3rd ed. problem 13-12) Repeat the calculation of the Fermi energy for a system of N noninteracting electrons in a cubic box of side L, but now treating the electrons as massless particles, so that $E = pc$ (this approximation holds when the kinetic energy of the electrons is much greater than their rest-mass energy). Integrate to get the total energy $E_{\rm tot}$ and compute the degeneracy pressure $p_{\rm deg}$.

   (Hint: all the steps are the same as in Sections 13-5 and 13-6 of the textbook, except that the energy of the single-particle state has a different form than Eq. (13-54). You should find $p_{\rm deg} \propto V^{-4/3}$ for a relativistic electron gas; such a system has no equilibrium size that balances the gravitational and degenerate-electron pressures.)

continued....

4. Consider N spinless particles in a 3-dimensional infinite square well with sides of length L.

   (a) Compute the total energy of the ground state of this system of N particles, assuming that the particles are distinguishable. What is the total energy of the ground state if the particles are identical bosons?

   (b) Compute the total energy of the first excited state of the system. What is the degeneracy of this state for the system of N distinguishable particles? What is the degeneracy if the N particles are identical bosons?

   (c) At finite temperature T, the probability of a system being in a particular quantum state with total energy E is $e^{-E/k_B T}$ (divided by a normalization factor), where k_B is Boltzmann's constant. Compute the relative probability $P(\text{1st excited})/P(\text{ground state})$ for the system of N distinguishable particles and for the system of N identical bosons. *(Note: this statistical effect is why noninteracting bosons seem to "enjoy" all being in the same state even more than distinguishable particles do.)*