Formula sheet for midterm exam *This page will be provided with the exam.*

Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = H\Psi(\vec{r},t) \tag{1}$$

Energy eigenstates: $H = \vec{p}^2/2m + V(\vec{r})$

$$H\psi(\vec{r}) = E\psi(\vec{r}), \qquad \Psi(\vec{r},t) = \psi(\vec{r})\,e^{-iEt/\hbar} \tag{2}$$

Differential operators corresponding to momentum and energy:

$$\vec{p} \to -i\hbar\vec{\nabla} \qquad E \to i\hbar\frac{\partial}{\partial t} \tag{3}$$

Harmonic oscillator in 1 dimension:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2, \qquad H|n\rangle = E_n|n\rangle, \qquad E_n = \hbar\omega(n+1/2) \tag{4}$$

Raising and lowering operators:

$$a|n\rangle = \sqrt{n}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \tag{5}$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger), \qquad p_x = -i\sqrt{\frac{m\omega\hbar}{2}}(a-a^\dagger) \tag{6}$$

Harmonic oscillator in 3 dimensions (isotropic), $H = \vec{p}^2/2m + m\omega^2\vec{r}^2/2 = H_x + H_y + H_z$: spatial wavefunctions are just the products of the solutions of the 1-dim harmonic oscillator.

$$H|n_1n_2n_3\rangle = E_{n_1n_2n_3}|n_1n_2n_3\rangle, \qquad E_{n_1n_2n_3} = \hbar\omega(n_1+n_2+n_3+3/2); \tag{7}$$

the wavefunctions can also be written in terms of the spherical harmonics,

$$\langle\vec{r}|n_r\ell m\rangle = R_{n_r\ell}(r)Y_{\ell m}(\theta,\phi), \qquad E_{n_r\ell m} = \hbar\omega(2n_r+\ell+3/2). \tag{8}$$

Particle in a box (infinite square well), 1 dimensional:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}, \qquad 0 \le x \le L, \qquad E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2 \tag{9}$$

Particle in a box, 3 dimensional, $L\times L\times L$, one corner at the origin:

$$\psi(x,y,z) = \left(\frac{2}{L}\right)^{3/2}\sin\frac{n_1\pi x}{L}\sin\frac{n_2\pi y}{L}\sin\frac{n_3\pi z}{L}, \qquad E = \frac{\hbar^2\pi^2}{2mL^2}(n_1^2+n_2^2+n_3^2) \tag{10}$$

Angular momentum:

$$L^2|\ell,m\rangle = \hbar^2\ell(\ell+1)|\ell,m\rangle, \qquad L_z|\ell,m\rangle = \hbar m|\ell,m\rangle, \qquad m_{\max} = \ell \tag{11}$$

$$L_\pm = L_x \pm iL_y, \qquad [L^2, L_\pm] = 0, \qquad L_\pm|\ell,m\rangle = C_\pm(\ell,m)|\ell,m\pm1\rangle \tag{12}$$

$$C_+(\ell,m) = \hbar\sqrt{(\ell-m)(\ell+m+1)}, \qquad C_-(\ell,m) = \hbar\sqrt{(\ell+m)(\ell-m+1)} \tag{13}$$

$$\vec{L} = \vec{r}\times\vec{p} \qquad \text{so, e.g., } L_z = xp_y - yp_x. \tag{14}$$

Addition of angular momentum: when $\vec{J} = \vec{L} + \vec{S}$,

$$j = \ell+s,\ldots,|\ell-s|, \qquad J_z = L_z + S_z, \qquad J_\pm = L_\pm + S_\pm \tag{15}$$

Spherical harmonics, up to $\ell = 2$:

$$\begin{aligned} Y_{00} &= \frac{1}{\sqrt{4\pi}} \qquad Y_{\ell m}(-\hat{r}) = (-1)^\ell Y_{\ell m}(\hat{r}) \\ Y_{11} &= -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \qquad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y_{1,-1} = \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi} \\ Y_{22} &= \sqrt{\frac{15}{32\pi}} \sin^2\theta e^{2i\phi} \qquad Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta e^{i\phi} \qquad Y_{20} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1) \\ & Y_{2,-1} = \sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta e^{-i\phi} \qquad Y_{2,-2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta e^{-2i\phi} \end{aligned} \tag{16}$$

Pauli principle: under the exchange of any two identical particles, multi-particle wavefunctions are symmetric for bosons (integer spin), antisymmetric for fermions (half-odd-integer spin).
Time-independent perturbation theory:

$$H = H_0 + \lambda H_1, \qquad H_0|\phi_n\rangle = E_n^{(0)}|\phi_n\rangle, \qquad H|\psi_n\rangle = E_n|\psi_n\rangle \tag{17}$$

$$E_n = E_n^{(0)} + E_n^{(1)} + E_n^{(2)} + \cdots \tag{18}$$

$$E_n^{(1)} = \langle\phi_n|\lambda H_1|\phi_n\rangle \qquad E_n^{(2)} = \sum_{k\neq n} \frac{|\langle\phi_n|\lambda H_1|\phi_k\rangle|^2}{E_n^{(0)} - E_k^{(0)}} \tag{19}$$

$$|\psi_n\rangle = |\phi_n\rangle + \sum_{k\neq n} \frac{\langle\phi_k|\lambda H_1|\phi_n\rangle}{E_n^{(0)} - E_k^{(0)}}|\phi_k\rangle + \mathcal{O}(\lambda^2) \tag{20}$$

Degenerate perturbation theory: first find the combinations of states that diagonalize the matrix $\langle\phi_i|\lambda H_1|\phi_j\rangle$ made up of states degenerate at zeroth order. The eigenvalues of this matrix are $E_n^{(1)}$.
Time-dependent perturbation theory:

$$H_0|\phi_n\rangle = E_n^{(0)}|\phi_n\rangle, \qquad i\hbar\frac{d}{dt}|\psi(t)\rangle = (H_0 + \lambda V(t))|\psi(t)\rangle \tag{21}$$

$$|\psi(t)\rangle = \sum_n c_n(t) e^{-iE_n^{(0)}t/\hbar}|\phi_n\rangle \tag{22}$$

For $c_k(0) = 1$ and all other $c_n(0) = 0$, to first order in λ the coefficients are

$$c_m(t) = \frac{1}{i\hbar}\int_0^t dt' e^{i\omega_{mk}t'}\langle\phi_m|\lambda V(t')|\phi_k\rangle \tag{23}$$

where $\omega_{mk} = (E_m^{(0)} - E_k^{(0)})/\hbar$. The transition probability is $P_{k\to m}(t) = |c_m(t)|^2$.
Variational principle: for *any* wavefunction $|\Psi\rangle$, $\langle\Psi|H|\Psi\rangle \geq E_0 =$ ground state energy of H.
Some math:
Taylor series about $x = 0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=0} x^n \tag{24}$$

Eigenvalues of a matrix M (I is the unit matrix; solve for λ):

$$\det(M - \lambda \cdot I) = 0 \tag{25}$$

The table of Clebsch-Gordan coefficients from the Review of Particle Physics will also be provided.