Carleton University Physics Department
PHYS 4708 (Winter 2016, H. Logan)
Homework assignment #3

Handed out Mon. Feb. 1; due Wed. Feb. 10, 2016 at the start of class.
Problems are worth 5 points each unless noted otherwise.

1. *[10 points]* (similar to Gasiorowicz 3rd ed. problem 11-11) Consider a two-dimensional harmonic oscillator with unperturbed Hamiltonian

$$H_0 = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2), \tag{1}$$

with eigenstates $|n_x n_y\rangle$ whose energies are $E^{(0)}_{n_x n_y} = \hbar\omega(n_x + n_y + 1)$. It is subject to a perturbing Hamiltonian

$$H_1 = 2\lambda xy. \tag{2}$$

(a) Compute the first- and second-order energy shifts $E^{(1)}_{00}$ and $E^{(2)}_{00}$ of the ground state of this harmonic oscillator. You can use the expression for x in terms of raising and lowering operators,

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_x + a_x^\dagger), \tag{3}$$

where $a_x|n_x\rangle = \sqrt{n_x}|n_x - 1\rangle$ and $a_x^\dagger|n_x\rangle = \sqrt{n_x + 1}|n_x + 1\rangle$ and the analogous expression for y.

(b) Compute the first-order energy shifts of the two states that make up the first excited energy level of the original Hamiltonian. *(Here you'll need to use degenerate perturbation theory.)*

(c) This problem can be solved exactly by writing

$$H = H_0 + H_1 = \frac{1}{2m}(p_{x'}^2 + p_{y'}^2) + \frac{1}{2}m\omega_x'^2 x'^2 + \frac{1}{2}m\omega_y'^2 y'^2, \tag{4}$$

with $x' = (x + y)/\sqrt{2}$ and $y' = (x - y)/\sqrt{2}$. Find $\omega'_{x,y}$ in terms of ω and λ and write down the exact perturbed ground-state energy E'_{00}. Do a series expansion of $\omega'_{x,y}$ out to second order in λ and check that these terms agree with your results for $E^{(1)}_{00}$ and $E^{(2)}_{00}$ found in part (a). Do the same for the energies E'_{10} and E'_{01} of the first excited states and compare with your results from part (b).

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2. Compute the energy shift from the anomalous Zeeman perturbation,

$$\Delta E_B = \frac{eB}{2m_e}\langle\phi_{\ell j m_j}|J_z + S_z|\phi_{\ell j m_j}\rangle, \tag{5}$$

for the $n = 2$ states of the hydrogen atom using the "brute force" method of expressing the eigenstates $|\phi_{\ell j m_j}\rangle$ as a linear combination of $|m_\ell m_s\rangle$ states in order to evaluate the matrix elements of S_z. (Use the table of Clebsch-Gordan coefficients to get the appropriate linear combinations.) Compare your results to the shifts found using Eq. (12-24) of the textbook,

$$\Delta E_B = \frac{eB\hbar}{2m_e} m_j \left(1 + \frac{j(j+1) - \ell(\ell+1) + 3/4}{2j(j+1)}\right). \tag{6}$$

3. Work out the energies for the Lyman-α transition spectral lines in hydrogen (relative to the zeroth-order energy $\Delta E_{2\to1} = (3/4)E_1^{\text{Bohr}}$) for the three following situations in the presence of an external magnetic field $\vec{B} = B\hat{z}$:

   (a) A fake situation in which the electron has no spin.

   (b) The case that the splitting due to the external B-field is very large compared to the fine structure, so that the fine structure can be ignored (work in the eigenbasis of definite m_ℓ and m_s).

   (c) The anomalous Zeeman case in which fine structure is a larger effect than the Zeeman splitting (feel free to use the final results for the energy shifts from the text or lecture).

   Note: Remember that this electric dipole transition obeys the selection rules $\Delta\ell = \pm1$, $\Delta m_\ell = 0, \pm1$ *and that an electric dipole transition cannot flip the spin of the electron.*

4. (Gasiorowicz 3rd edition problem 12-5) Consider a harmonic oscillator in three dimensions. If the relativistic expression for the kinetic energy is used, what is the shift in the ground-state energy?

   Hint: treat the relativistic correction as a small perturbation as we did for the hydrogen atom. Because the harmonic oscillator Hamiltonian can be written as $H_0 = |\vec{p}|^2/2m + m\omega^2 r^2/2 = |\vec{p}|^2/2m + V(x) + V(y) + V(z)$, *where* $V(x) = m\omega^2 x^2/2$, *etc., the three-dimensional harmonic oscillator can be solved by separation of variables in Cartesian coordinates with the result*

$$\psi_E(x, y, z) = u_{n_1}(x)u_{n_2}(y)u_{n_3}(z), \tag{7}$$

   where the functions u *are the 1-dimensional harmonic oscillator solutions in each direction. The corresponding energy eigenvalues are* $E = \hbar\omega(n_1 + n_2 + n_3 + 3/2)$.