PHYS 3308 — Handout on Kronecker Delta and Levi-Civita tensor and vector calculus

Cartesian coordinates: $x, y, z \rightarrow x_i, \quad i = 1, 2, 3$

Kronecker delta: $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Levi-Civita tensor:
(totally antisymmetric)
$$\epsilon_{ijk} = \begin{cases} +1 & \text{for } ijk = 123,\ 231, \text{ or } 312 \\ -1 & \text{for } ijk = 132,\ 321, \text{ or } 213 \\ 0 & \text{otherwise} \end{cases}$$

Repeated-index notation: $A_i B_i \equiv A_1 B_1 + A_2 B_2 + A_3 B_3$

Dot product: $\vec{A} \cdot \vec{B} = A_i B_i = A_i B_j \delta_{ij}$ (i and j implicitly summed)

Cross product: $(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k$

↑ i^{th} component of the vector $\vec{A} \times \vec{B}$

A useful identity:

$$\epsilon_{kij}\,\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

Vector derivatives in Cartesian coordinates:

$$(\vec{\nabla} T)_i = \frac{\partial T}{\partial x_i}$$ (i^{th} component) or $\vec{\nabla} T = \frac{\partial T}{\partial x_i} \hat{x}_i$

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial}{\partial x_i} A_i$$ (sum over i, as usual)

$$(\vec{\nabla} \times \vec{A})_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} A_k$$

Ch. 1 - Vector Analysis

- Vector notation: $\vec{A}$ (I will always use vector symbol on handwritten vectors. Griffiths uses bold face to denote the difference. I don't need to remind you to be careful with this in your notes + homework assignments.)

1.1 • Vector products

1) Scalar product = dot product

$\vec{A} \cdot \vec{B} = A\,B\,\cos\theta$ where θ is the angle between $\vec{A}$ + $\vec{B}$.

- More notation: $A \equiv |\vec{A}|$ = magnitude of $\vec{A}$

$\vec{A} \cdot \vec{A} = A^2$; if $\vec{A} \perp \vec{B}$ then $\vec{A} \cdot \vec{B} = 0$.

It is commutative: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

and distributive: $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

2) Cross product = vector product

$\vec{A} \times \vec{B} = A\,B\,\sin\theta\;\hat{n}$, where $\hat{n} \perp \vec{A}$ and $\vec{B}$, direction by right-hand rule.

- More notation: unit vector

$\hat{n}$ ("n-hat"): $|\hat{n}| = 1$.

It is not commutative: $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

It is distributive: $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

$\vec{A} \times \vec{A} = 0$.

3) Vector algebra in component form

Let's consider Cartesian coordinates.

The unit vectors are $\hat{x}$, $\hat{y}$, $\hat{z}$ (sometimes called $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$) parallel to the x, y, + z axes.

This is an orthonormal (mutually orthogonal, normalized) set of basis vectors that spans our 3-dim vector space. (Linear algebra.)

Any vector can be expressed in component form:

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

– More notation: A_x = x component of $\vec{A}$.

Finding the components: note that

$$\vec{A} \cdot \hat{x} = A_x \underbrace{\hat{x} \cdot \hat{x}}_{1} + A_y \underbrace{\hat{y} \cdot \hat{x}}_{0} + A_z \underbrace{\hat{z} \cdot \hat{x}}_{0}$$

$$= A_x \qquad (\text{+ likewise: } \vec{A} \cdot \hat{y} = A_y, \ \vec{A} \cdot \hat{z} = A_z)$$

Addition: $\vec{A} + \vec{B} = (A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) + (B_x \hat{x} + B_y \hat{y} + B_z \hat{z})$

$$= (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y} + (A_z + B_z)\hat{z}$$

Dot product: $\vec{A} \cdot \vec{B} = (A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) \cdot (B_x \hat{x} + B_y \hat{y} + B_z \hat{z})$

use $\hat{x} \cdot \hat{x} = 1$, $\hat{x} \cdot \hat{y} = 0$, etc

$$= A_x B_x + A_y B_y + A_z B_z$$

This also gives $\vec{A} \cdot \vec{A} = A_x^2 + A_y^2 + A_z^2 = A^2$

Cross product: first consider unit vectors.

$$\hat{x} \times \hat{x} = 0 = \hat{y} \times \hat{y} = \hat{z} \times \hat{z}$$

$$\hat{x} \times \hat{y} = \hat{z} \qquad \text{(RH rule)}$$
$$\hat{y} \times \hat{z} = \hat{x}$$
$$\hat{z} \times \hat{x} = \hat{y} \qquad \text{(note cyclicity)}$$

— make sure you can do this with your hand: easier than memorizing.

Reverse order picks up a minus sign:

$$\hat{y} \times \hat{x} = -\hat{z}$$
$$\hat{z} \times \hat{y} = -\hat{x}$$
$$\hat{x} \times \hat{z} = -\hat{y}$$

Thus,

$$\vec{A} \times \vec{B} = (A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) \times (B_x \hat{x} + B_y \hat{y} + B_z \hat{z})$$
$$= (A_y B_z - A_z B_y)\hat{x} + (A_z B_x - A_x B_z)\hat{y} + (A_x B_y - A_y B_x)\hat{z}$$

"Determinant form":

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

please use this when computing cross products until you're <u>really</u> used to it.

Levi-Civita tensor form: $(x, y, z) \to (1, 2, 3)$

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) = (1,2,3) \text{ and any cyclic permutation } (2,3,1), (3,1,2) \\ -1 & \text{if } (i,j,k) = (1,3,2), (3,2,1), (2,1,3) \\ 0 & \text{otherwise if } i=j \text{ or } i=k \text{ or } j=k \end{cases}$$

ϵ_{ijk} = totally antisymmetric tensor with $\epsilon_{123} = 1$.

$\longrightarrow$

Using this,

$$\left(\vec{A}\times\vec{B}\right)_i = \epsilon_{ijk} A_j B_k$$

w/ implicit summation over all repeated indices (summed over j and k.)

Notice the free index i: that indicates the result is a vector (3 components).

4) Triple products

$\vec{B}\times\vec{C}$ is a vector. We can dot another vector into it and make a scalar: Scalar triple product:

$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)$$

In Levi-Civita symbol language this is:

$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right) = A_i\left(\vec{B}\times\vec{C}\right)_i$$ (summed over i implicitly)

$$= A_i\,\epsilon_{ijk}\,B_j\,C_k$$ (implicit summation on all repeated indices)

– Note there are no free (unsummed) indices: this is a scalar.

The antisymmetric property of ϵ_{ijk} gives

$$A_i\,\epsilon_{ijk}\,B_j\,C_k = B_j\,\epsilon_{jki}\,C_k\,A_i = C_k\,\epsilon_{kij}\,A_i\,B_j$$ note cyclic ordering

$$= -A_i\,\epsilon_{ikj}\,C_k\,B_j = -B_j\,\epsilon_{jik}\,A_i\,C_k = -C_k\,\epsilon_{kji}\,B_j\,A_i$$

Translating back into vector product form, this gives the identities

$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right) = \vec{B}\cdot\left(\vec{C}\times\vec{A}\right) = \vec{C}\cdot\left(\vec{A}\times\vec{B}\right)$$ note cyclic ordering

$$= -\vec{A}\cdot\left(\vec{C}\times\vec{B}\right) = -\vec{B}\cdot\left(\vec{A}\times\vec{C}\right) = -\vec{C}\cdot\left(\vec{B}\times\vec{A}\right)$$

This is derived geometrically in the textbook (sec. 1.1.3).

We can also make a <u>vector triple product</u>:

$$\vec{A} \times (\vec{B} \times \vec{C})$$

To derive some identities, again let's use the Levi-Civita symbol:

$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = \epsilon_{ijk} A_j (\vec{B} \times \vec{C})_k$$

$$= \epsilon_{ijk} A_j \epsilon_{kmn} B_m C_n$$

$$= \epsilon_{kij} \epsilon_{kmn} A_j B_m C_n$$

recall implicit summation

* Note that the placement of parentheses <u>matters</u> a <u>lot</u>!

To simplify this, we can use

$$\underbrace{\epsilon_{kij} \epsilon_{kmn}}_{\text{sum on } k} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}$$

"Homework": convince yourself that this is correct.

where $\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$ is the Kronecker delta.

Also note that δ_{ij} provides another way to write a dot product: $\vec{A} \cdot \vec{B} = A_i B_i = A_i B_j \delta_{ij}$ (implicit sums)

We have,

$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = \epsilon_{kij} \epsilon_{kmn} A_j B_m C_n$$

$$= (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) A_j B_m C_n$$

$$= B_i \vec{A} \cdot \vec{C} - C_i \vec{A} \cdot \vec{B}$$

Going back to vector notation, we've just derived the so-called BAC-CAB rule:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B})$$

This is very useful for simplifying complicated vector products.

5) Finally, a bit of notation:

a) Position vector (from the origin to point (x,y,z)):

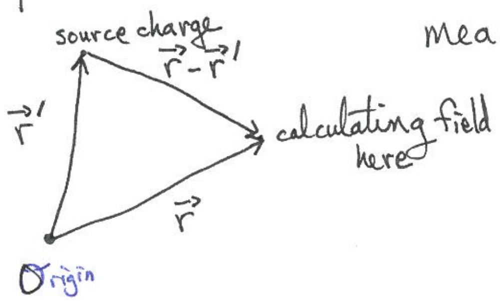
$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$r = \sqrt{x^2+y^2+z^2}$, $\hat{r} = \dfrac{\vec{r}}{r}$ (unit vector pointing radially outward)

b) Infinitesimal displacement vector (from (x,y,z) to $(x+dx, y+dy, z+dz)$):

$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$ (could call this $d\vec{r}$, but choose to define a separate letter for later clarity)

c) Separation vector (from a source point $\vec{r}'$ to a measurement point $\vec{r}$):



Notation: $\vec{\mathscr{r}} \equiv \vec{r} - \vec{r}'$ I will follow Griffith's usage.

$$\mathscr{r} = |\vec{r}-\vec{r}'|$$

$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}$$

In cartesian coords,

$$\vec{\mathscr{r}} = (x-x')\,\hat{x} + (y-y')\,\hat{y} + (z-z')\,\hat{z}.$$

1.4 Curvilinear coordinates: introduction

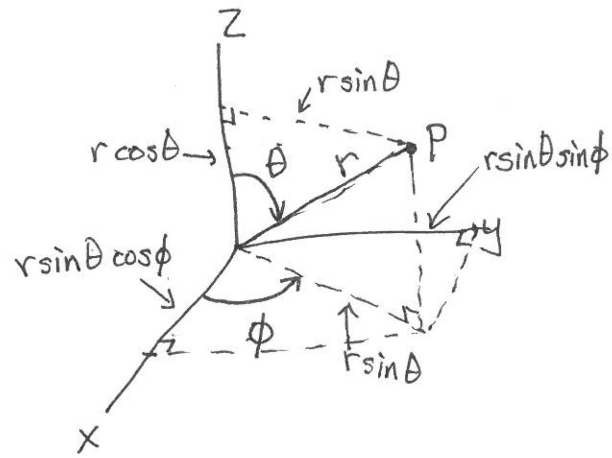
We will work with two non-Cartesian coordinate systems:

a) spherical coordinates (= spherical polar coordinates) and

b) cylindrical coordinates.

Let's now set up the notation and review the basics.

a) Spherical coordinates



Coordinates are r, θ, ϕ.

$r : [0, \infty)$ distance from origin

$\theta : [0, \pi]$ polar angle

$\phi : [0, 2\pi)$ azimuthal angle

Relation to cartesian coords: easy to derive from the figure and a bit of trigonometry.

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

"HW": derive for yourself the inverse formulas for r, θ, ϕ in terms of x, y, z. Answers are inside the back cover of textbook.

Unit vectors: $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$. Point in direction of increase of corresponding coordinate. They are orthonormal, but note that their direction at point P <u>changes</u> as P moves around in space (!).

→

$\hat{r}$: points radially outward.

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

(direction depends on particular values of θ and ϕ)

"HW": check that $|\hat{r}| = 1$.

$\hat{\theta}$: points $\perp \hat{r}$, in dir. of increasing θ — in the plane spanned by $\vec{r}$ and $\hat{z}$.

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

"HW": check that $|\hat{\theta}| = 1$ and $\hat{r}\cdot\hat{\theta} = 0$.

$\hat{\phi}$: points $\perp \hat{r}$ and $\perp \hat{\theta}$, <u>in the x-y plane</u>, in dir. of increasing ϕ:

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}.$$

"HW": check that $|\hat{\phi}| = 1$, and $\hat{\phi}\cdot\hat{r} = \hat{\phi}\cdot\hat{\theta} = 0$.

Note that this forms a right-handed coordinate system:

$$\hat{r}\times\hat{\theta} = \hat{\phi}$$
$$\hat{\theta}\times\hat{\phi} = \hat{r}$$
$$\hat{\phi}\times\hat{r} = \hat{\theta}$$

Infinitesimal displacement:

$$dl_r = dr$$
$$dl_\theta = r\,d\theta \quad \text{(must have units of length!)}$$
$$dl_\phi = r\sin\theta\,d\phi$$

$\longrightarrow$

so $d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

(compare $d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$)

Note also infinitesimal volume element:

$$d\tau = dl_r\,dl_\theta\,dl_\phi = dr \cdot r\,d\theta \cdot r\sin\theta\,d\phi$$
$$= r^2 \sin\theta\,dr\,d\theta\,d\phi$$

familiar from integrals in spherical coordinates.

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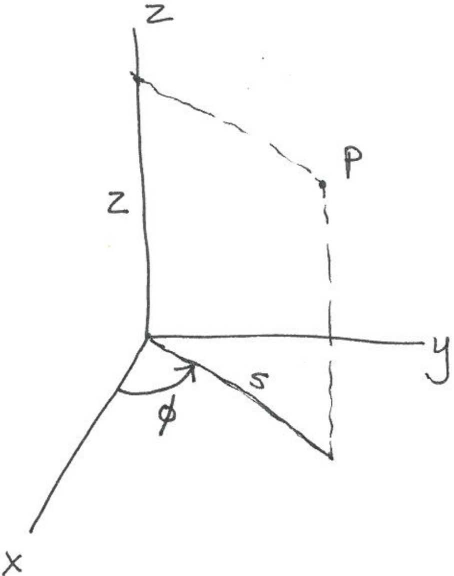
b) <u>Cylindrical coordinates</u>



Coordinates are s, ϕ, z

usually called r or ρ, but I'm following text to keep this distinct from spherical coords. (and ρ = volume charge density)

$s: [0, \infty)$
$\phi: [0, 2\pi)$
$z: (-\infty, \infty)$

$x = s\cos\phi$, $y = s\sin\phi$, $z = z$

Unit vectors:

$\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

points outward in x-y plane. Direction depends on ϕ.

$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

$\hat{z} = \hat{z}$

Again, $\hat{s} \times \hat{\phi} = \hat{z}$, $\hat{\phi} \times \hat{z} = \hat{s}$, $\hat{z} \times \hat{s} = \hat{\phi}$.

Infinitesimal displacement:

$$dl_s = ds$$
$$dl_\phi = s\,d\phi$$
$$dl_z = dz$$

so $d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$

Infinitesimal volume element is

$$d\tau = dl_s\,dl_\phi\,dl_z = ds\,s\,d\phi\,dz = s\,ds\,d\phi\,dz$$

familiar from integrals in 2-dim polar coords, with the dz tacked on.

A.2,3 Note that in all 3 of these coordinate systems, we can write the infinitesimal displacement $(u, v, w) \to (u+du, v+dv, w+dw)$ as:

$$d\vec{l} = f\,du\,\hat{u} + g\,dv\,\hat{v} + h\,dw\,\hat{w}$$

where f, g, h are some functions of the generalized coordinates $u, v, + w$. The unit vectors $\hat{u}, \hat{v}, \hat{w}$ are in general also functions of position.

	u	v	w	f	g	h
Cartesian	x	y	z	1	1	1
Spherical	r	θ	ϕ	1	r	$r\sin\theta$
Cylindrical	s	ϕ	z	1	s	1

We'll use these next to review the vector calculus we'll need: gradient, divergence, and curl.

Recap: $d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$ Cartesian

$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$ Spherical

$d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$ Cylindrical