PHYS 5802
GROUP THEORY AND ITS APPLICATION TO PARTICLE PHYSICS

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Motivation to Study Group Theory

Study of the symmetries respected by physical systems gives us insight into those systems. Symmetries arise as a consequence of unmeasurability (non-observability) or indistinguishability in nature. For instance, if we say that space is homogeneous or, in other words, that we cannot measure an absolute position in space, this is a symmetry. The system is invariant under the symmetry transformation of spatial translation.

Group theory is the study of symmetry transformations that have particular "group properties". The consequences of such symmetries can be profound. For the example of invariance under spatial transformation, the consequence is conservation of linear momemtum.

Group theory is important because symmetries alone give useful information about a physical system, quite apart from the details of that system. The existence of symmetries allows one to draw general conclusions.

Consequences of symmetries include:

- conservation laws
- identification of conserved quantum numbers (useful for symmetry labels and for understanding energy degeneracy)
- selection rules
- relations between matrix elements for observables
- simplification

Stancu makes a useful breakdown of various types of symmetries as follows.

1. Discrete permutation symmetries
2. Continuous space-time symmetries
3. Discrete space-time symmetries
4. Continuous internal symmetries
5. Discrete internal symmetries

1. Discrete permutation symmetries

In quantum systems, we expect physical observables to be invariant under the exchange - permutation - of identical particles. Such transformations constitute the permutation or symmetric group $\mathcal{S}_n$. The indistinguishability of identical particles, as realized via this permutation symmetry, leads to the formulation of Bose-Einstein and Fermi-Dirac statistics. Also, the symmetric group is related to some other important continuous groups. Consequently, there is a further practical advantage to studying its properties.

2. Continuous space-time symmetries

These are probably most familiar to us - from classical physics and extended to quantum physics. The transformations include:

- Spatial translations, $\vec{x} \longrightarrow \vec{x}' = \vec{x} + \vec{a}$, based on the assumption of the homogeneity of space.

This is valid for an isolated system (with no external field) such that the potential describing the system does not depend on the position of the origin of coordinates. The consequence is the conservation of linear momentum.

• Rotations in three dimensions, $x_i \longrightarrow x'_i = R_{ij}x_j$ with $i, j = 1, 2, 3$ and R_{ij} a length preserving rotation matrix. This arises from the assumption of spatial isotropy - no preferred direction - and is again valid for isolated systems. The properties of a system obeying this symmetry do not depend on orientation in space. The consequence is conservation of angular momentum.

• Time translation, $t \longrightarrow t' = t + t_0$, arises from the assumption of homogeneity of time - no absolute time. Given the same physical conditions, the same phenomena can be reproduced at any time. This is valid for conservative systems that are isolated or subject to a time-independent external field. (The Lagrangian or Hamiltonian describing the system has no explicit time dependence.). The consequence is conservation of energy.

• Lorentz transformations, $x^{\mu} \longrightarrow x'^{\mu} = \Lambda^{\mu}_{\nu} x_{\nu}$, include rotations as above and boosts such that systems moving with respect to each other measure the same velocity of light. Valid in special relativity for inertial reference frames.

3. Discrete space-time symmetries

Examples of these symmetries include:

• Spatial Periodicity, such that, for instance, a crystal lattice is invariant under translations $\vec{x} \longrightarrow \vec{x}' = \vec{x} + \vec{A}$ where $\vec{A}$ is a constant rather than a continuous variable. This type of symmetry extends to the point groups of solid state physics.

• Spatial inversion or reflection invariance arises from indistinguishability of left and right in a system. The parity operator $\mathcal{P}$ acts as $\mathcal{P}\vec{x} = \vec{x}' = -\vec{x}$. Operating twice with $\mathcal{P}$ restores the original system. A consequence of this symmetry is that quantum mechanical states can be characterized by even and odd wave functions.

• Time reversal is the operation associated with the unmeasurability of the direction of the flow of time, such that $\mathcal{T}t = t' = -t$.

4. Continuous internal symmetries

These more abstract symmetries act in the space of internal degrees of freedom. In particle physics, these include isospin, hypercharge, color, and flavor symmetries as well as gauge symmetries. They can lead to the conservation of “generalized charges”, including actual electromagnetic charge, lepton number, etc.

5. Discrete internal symmetries

In particle physics, these include the charge conjugation symmetry $\mathcal{C}$, transforming particle to anti-particle and $\mathcal{G}$-parity, which consists of charge conjugation followed by a rotation through π in isospin space.

- Not all the above symmetries are exact.
- Even approximate symmetries can be useful.
- Symmetries are spontaneously broken when the vacuum does not respect the overall symmetry of the system.
- Continuous symmetries are defined in terms of continuous parameters:

  Global - independent of space-time

  Local - space-time dependent - further consequences arise giving system dynamics

The Consequences of Symmetry

Consider two simple classical systems in Newtonian framework, followed more generally with a Lagrangian/Hamiltonian formulation.

1. Isolated system (not subject to external force) of 2 particles

- Assume absolute position is not measurable

- Relative to origin $\mathcal{O}$, position of particle i labeled by $\vec{r}_i$

- Origin $\mathcal{O}'$ located at $-\vec{a}$ relative to $\mathcal{O}$, such that $\vec{r'}_i = \vec{r}_i + \vec{a}$

- The potential describing the interaction of the two particles must satisfy

$$V(\vec{r}_1, \vec{r}_2) = V(\vec{r}_1 + \vec{a}, \vec{r}_2 + \vec{a})$$

$$\Rightarrow V = V(\vec{r}_1 - \vec{r}_2)$$

- With the notation that $\vec{\nabla}_i$ represents the gradient with respect to the coordinates of particle i:

$$\vec{F}_i = -\vec{\nabla}_i V$$

- Thus, the total force

$$\vec{F} = \vec{F}_1 + \vec{F}_2 = -\vec{\nabla}_1 V - \vec{\nabla}_2 V = 0$$

- But the total force is

$$\vec{F} = \frac{d\vec{p}}{dt} = 0$$

$\Rightarrow \vec{p}$ is constant

This demonstrates the line of reasoning: an assumption of unmeasurability of absolute position expressed as an invariance under spatial translation yields conservation of linear momentum.

2. Single particle in rotationally symmetric potential

- Assume a 2-d system is rotationally symmetric

- Single particle of mass m in 2-d potential

$$\Rightarrow V(r,\theta) = V(r)$$

$$\frac{\partial V}{\partial \theta} = 0 = \frac{\partial V}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial \theta}$$

- Using the relation between (x,y) and (r,θ) and the equations of motion $m\ddot{x} = -\frac{\partial V}{\partial x}$ and $m\ddot{y} = -\frac{\partial V}{\partial y}$,

$$\begin{aligned}\frac{\partial V}{\partial \theta} &= 0 = -y\frac{\partial V}{\partial x} + x\frac{\partial V}{\partial y} = y(m\ddot{x}) - x(m\ddot{y}) \\ &= -\frac{d}{dt}(\vec{r}\times\vec{p})_z = -\frac{d}{dt}L_z = 0\end{aligned}$$

$\Rightarrow$ Angular momentum is conserved.

We see the same line of reasoning followed.

3. Lagrangian formulation

Consider a Lagrangian that is a function of the generalized coordinates q_i, the generalized velocities $\dot{q}_i$, and possibly time t - $L(q_i, \dot{q}_i, t)$.

- Assuming Hamilton's principle $\delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt = 0$ leads to Euler-Lagrange equations:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0$$

- Assuming a symmetry such that L is independent of q_i (invariance under "translation") yields

$$\frac{\partial L}{\partial q_i} = 0 = \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})$$

$\Rightarrow p_i = \frac{\partial L}{\partial \dot{q}_i} =$ constant

These generalized (or conjugate) momenta are conserved

Invariance under translation in the generalized coordinate q_i implies conservation of the corresponding generalized momentum $p_i = \frac{\partial L}{\partial \dot{q}_i}$.

Consider next the assumption of homogeneity of time which implies the Lagrangian has no explicit time dependence, $\frac{\partial L}{\partial t} = 0$. Thus

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i = \frac{d}{dt}(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i})$$

where the Euler-Lagrange equations have been used in the last step. Thus

$$\frac{d}{dt}(L - \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}) = 0$$

Since $L = T - V$, with $T(V)$ the kinetic (potential) energy and since, in a scleronomic system (that is, one with no explicit time dependence in the transformation equations $x_{\alpha,i} = x_{\alpha,i}(q_j, t)$ with α a particle label) the second term above is $2T$ (Euler's theorem), we have

$\Rightarrow \frac{d}{dt}(T + V) = 0$

Energy conservation arises as a consequence of the assumption of homogeneity of time.

Unmeasurability of	invariance under	conservation of
Absolute spatial position	spatial translation	linear momentum
Absolute time	time translation	energy
Absolute spatial direction	spatial rotation	angular momentum

Express the connection between invariance principle and conservation law via classical Poisson brackets

For two continuous functions of generalized coordinates and momenta, $f(q_i, p_i)$ and $h(q_i, p_i)$

$$\{f, h\} \equiv \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial h}{\partial q_i}\right)$$

If the Poisson bracket vanishes, we say f and h commute.

Consider the Hamiltonian $H \equiv T + V = H(q_i, p_i)$. Hamilton's equations of motion are

$\dot{p}_i = -\frac{\partial H}{\partial q_i}$ and $\dot{q}_i = \frac{\partial H}{\partial p_i}$

For $F(q_i, p_i)$ a dynamical variable with no explicit time dependence

$$\begin{aligned} \frac{dF}{dt} = \dot{F} &= \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i \\ &= \frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} = \{F, H\} \end{aligned}$$

$\{F, H\} = 0$ implies that F is a constant of the motion.

Symmetries in quantum physics

The classical Poisson bracket relation $\dot{F} = \{F, H\}$ corresponds, for A an operator and H the quantum mechanical Hamiltonian, to the quantum relation

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{1}{i\hbar}[A, H]$$

in the Heisenberg picture with time independent state vectors.

Thus if

1. A is Hermitian
2. A has no explicit time dependence and
3. A commutes with the Hamiltonian

$\Rightarrow$ A represents a conserved observable

$\frac{dA}{dt} = 0$ is equivalent to $\frac{d\langle A \rangle}{dt} = 0$ implying that the eigenvalues of A are constant.

We can develop this in an alternate way.

State ψ satisfies the Schrodinger equation $H\psi = i\hbar\frac{\partial\psi}{\partial t}$

The expectation value of an operator A is

$$\begin{aligned}\langle A\rangle &= (\psi, A\psi) = \langle\psi|A\psi\rangle \\ &= \int d^3x\psi^*(x,t)A\psi(x,t)\end{aligned}$$

So the rate of change of the expectation value is

$$\begin{aligned}\frac{d\langle A\rangle}{dt} &= (\frac{\partial\psi}{\partial t}, A\psi) + (\psi, A\frac{\partial\psi}{\partial t}) + (\psi, \frac{\partial A}{\partial t}\psi) \\ &= \frac{i}{\hbar}[(H\psi, A\psi) - (\psi, AH\psi)] + (\psi, \frac{\partial A}{\partial t}\psi) \\ &= \frac{i}{\hbar}(\psi, [H, A]\psi) + (\psi, \frac{\partial A}{\partial t}\psi)\end{aligned}$$

equivalent to the Heisenberg relation given above.

As usual in quantum mechanics, the eigenvalues of Hermitian operators that commute with the Hamiltonian are constants of the motion and can be specified simultaneously with the energy eigenvalues. This situation arises when a system is invariant under some symmetry.

Must express the transformation properties of states and operators under a symmetry

In quantum mechanics, an observable is $|\langle\varphi|\mathcal{O}|\psi\rangle|$ where $\mathcal{O}$ is a Hermitian operator.

For invariance under some symmetry transformation, observables must be unchanged under that transformation.

For $\mathcal{O} = 1$, the identity operator, the modulus of the scalar product $|\langle\varphi|\psi\rangle|$ is the observable. Under the symmetry transformation $\varphi \longrightarrow \mathcal{T}\varphi = \varphi'$, we must preserve orthonormality

$$|\langle\varphi|\psi\rangle| = |\langle\varphi'|\psi'\rangle|$$

This implies that the transformations $\mathcal{T}$ must be linear and unitary ($U^\dagger = U^{-1}$) or antilinear and anti-unitary.

Linear $\quad U(a\varphi + b\psi) = aU\varphi + bU\psi$

Anti-linear $\quad T(a\varphi + b\psi) = a^*T\varphi + b^*T\psi$

Anti-unitary operator ($\langle \mathcal{T}\varphi | \mathcal{T}\psi \rangle = \langle \psi | \varphi \rangle = \langle \varphi | \psi \rangle^*$) is anti-linear and norm-preserving.

So typically $\varphi \to U\varphi = \varphi'$, with U unitary.

The trivial result

$$| \langle \varphi | \mathcal{O} | \psi \rangle | = | \left\langle \varphi | U^\dagger U \mathcal{O} U^\dagger U | \psi \right\rangle | \equiv | \langle \varphi' | \mathcal{O}' | \psi' \rangle |$$

follows for any operator $\mathcal{O}$. That is, the matrix element is unchanged under the simultaneous transformation of the states $U\varphi = \varphi'$ and the operator $\mathcal{O}' = U\mathcal{O}U^\dagger$.

Invariance arguments arise if an operator $\mathcal{O}$ representing an observable is unchanged under

$$\mathcal{O}' = U\mathcal{O}U^\dagger = \mathcal{O}$$

That is, $[\mathcal{O}, U] = 0$.

Consider the case $\mathcal{O} = H$, the Hamiltonian. For $H' = H$ the dynamics of the system remain invariant under the transformation represented by U.

If $|\psi\rangle$ is an eigenstate of H with energy eigenvalue E,

$$H|\psi\rangle = E|\psi\rangle$$

then $U|\psi\rangle$ is also an eigenstate with the same eigenvalue. For $[H,U]=0$, U is a constant of the motion. If U is Hermitian it represents a conserved observable. If U is not Hermitian, there is a Hermitian operator associated with it that represents the conserved observable.

Some examples

1. Discrete space-time transformation: Parity

The Parity or spatial reflection operator is an example of an operator whose application twice yields the identity $\mathcal{P}^2 = 1$. $\mathcal{P}^{-1} = \mathcal{P}^{\dagger} = \mathcal{P}$ is a Hermitian operator with eigenvalues ± 1.

Suppose $\psi(\vec{r})$ is a non-degenerate eigenstate of some system corresponding to energy eigenvalue E. Suppose that the system is invariant under the Parity transformation.

$\Rightarrow \mathcal{P}\psi(\vec{r}) = \psi(-\vec{r})$ is also an eigenstate, with the same eigenvalue E.

Thus $\psi(\vec{r})$ and $\mathcal{P}\psi(\vec{r})$ cannot be linearly independent.

$\Rightarrow$ for all $\vec{r}$

$$\mathcal{P}\psi(\vec{r}) = \psi(-\vec{r}) = \pi\psi(\vec{r})$$

where π is a proportionality constant. The above is equally true for $-\vec{r}$

$$\mathcal{P}\psi(-\vec{r}) = \psi(\vec{r}) = \pi\psi(-\vec{r})$$

Applying $\mathcal{P}$ again to either equation above yields $\pi^2 = 1$ such that $\pi = \pm 1$.

Thus any nondegenerate eigenstate of a Parity-invariant system will obey either

$$\mathcal{P}\psi(\vec{r}) = +\psi(\vec{r})$$

or

$$\mathcal{P}\psi(\vec{r}) = -\psi(\vec{r})$$

The eigenstates are simultaneously eigenfunctions of the Parity operator and are either even or odd under spatial reflection.

This means we have selection rules for matrix elements of operators with well-defined parity.

For instance, electric monopole transitions between states of different parity are forbidden whereas, for a dipole operator, transitions between states of the same parity are forbidden.

2. Continuous space-time transformation: Translation

Assume a system is invariant under spatial translation in one dimension.

Consider that an observer using coordinate system S with origin O at $x = 0$ describes the system by

$$\psi(x) = Ne^{-x^2/x_0^2}.$$

This is a Gaussian centred at $x = 0$.

Consider another observer using coordinate system S' with origin O' displaced along the x-axis to $x = a$.

That is, $x' = x - a$.

We know how the observer using S' should describe this system: as a Gaussian centred at $x' = -a$.

$$\psi'(x') = Ne^{-(x'+a)^2/x_0^2} = \psi(x)$$

where the relation between x and x' was used in the second step.

We can rewrite as

$$\psi'(x) = Ne^{-(x+a)^2/x_0^2} = \psi(x + a)$$

This tells us the transformed form of the original function. We can see how this corresponds to an (unitary) operator acting on the original form as follows.

Assume first an infinitesmal translation through ϵ.

$$\begin{aligned} \psi'(x) &= \psi(x+\epsilon) \\ &\sim \psi(x) + \epsilon\frac{\partial\psi}{\partial x} \\ &= (1 + \frac{i\epsilon}{\hbar}p_x)\psi(x) \end{aligned}$$

where the quantum mechanical operator $p_x = -i\hbar\frac{\partial}{\partial x}$ has been identified.

We can formally build up the finite translation by successive application of infinitesmal translations

$$\begin{aligned} \psi'(x) &= \lim_{N\to\infty}(1 + \frac{a}{N}\frac{\partial}{\partial x})^N\psi(x) \\ &= \psi(x+a) = \sum_{n=0}^{\infty}\frac{a^n}{n!}\frac{d^n}{dx^n}\psi(x) \\ &= exp[ia(-i\partial/\partial x)]\psi(x) = D(a)\psi(x) \\ &= e^{iap_x/\hbar}\psi(x) \end{aligned}$$

Since p_x is Hermitian, the displacement operator $D(a) = e^{iap_x/\hbar}$ expressing the translation is unitary.

The momentum operator is called the infinitesmal generator of translations. It is the Hermitian operator representing the conserved observable (momentum) that corresponds to the invariance of the system under D, translations.

We can also see how the exponentiated form arises for the finite translation by recognizing that one translation, through ϵ, followed by another, through a, is again a translation, through $a + \epsilon$. Using the form for the infinitesmal translation

$$D(a)D(\epsilon) = D(a + \epsilon) = D(a)(1 + i\epsilon\frac{p_x}{\hbar})$$

we see the displacement operator satisfies the differential equation

$$\frac{dD(a)}{da} = \lim_{\epsilon \to 0} \frac{D(a+\epsilon) - D(a)}{\epsilon} = i\frac{p_x}{\hbar}D(a)$$

with the solution $D(a) = e^{iap_x/\hbar}$.

This generalizes to a translation in three dimensions as

$$D(\vec{a}) = D_x(a_x)D_y(a_y)D_z(a_z) = exp(i\vec{a} \cdot \vec{p}/\hbar)$$

The result $\psi'(x) = D(a)\psi(x)$ links back to our general transformation result $\varphi' = U\varphi$. We can also look at the simultaneous transformation of an operator in this particular case. Focus on the Hamiltonian because we will impose that it be invariant.

Observer S writes the time independent Schrodinger equation

$$H(x)\varphi(x) = E\varphi(x).$$

Observer S' writes the Schrodinger equation

$$H'(x')\varphi'(x') = E\varphi'(x').$$

These equations are true for all x so we can rewrite the latter as

$$H'(x)\varphi'(x) = E\varphi'(x).$$

Of course the Hamiltonian's energy eigenvalue E must be common to the two observers.

Operate with the displacement operator on Observer S's equation:

$$\begin{aligned} D(a)H(x)(D^\dagger(a)D(a))\varphi(x) &= ED(a)\varphi(x) \\ (D(a)H(x)D^\dagger(a))\varphi'(x) &= E\varphi'(x) \end{aligned}$$

So we can identify $H'(x) = D(a)H(x)D^\dagger(a)$, as expected. Because this system is assumed to be invariant under translation, we have $H'(x) = H(x)$ implying $[H(x), D(a)] = 0$ such that $[H(x), p_x] = 0$. Thus eigenfunctions of the Hamiltonian are simultaneously momentum eigenstates.

3. Discrete space-time transformation: Particle on a one-dimensional lattice

Again consider a system invariant under translation in one-dimension but this time assume a periodic potential $V(x) = V(x + na)$, where a is now a constant and n is an integer. Other than the periodicity, we do not specify the potential so our results arise only because of the symmetry. This sort of discrete translational symmetry is useful in, for instance, metals which generally have ions arranged in a crystal structure so that electrons see a periodic potential.

We have the operator representing a general displacement through $-a$ as $D(a) = e^{iap/\hbar}$. Let's write our displacement operator here as $T(n) = e^{-ina\hat{k}}$ where the operator $p/\hbar$ is replaced by the operator $\hat{k}$.

$$T(n)\,|x\rangle = |x + na\rangle$$

The label n gives the (discrete) amount of translation and we will use the label k as the eigenvalue of $\hat{k}$. Because the Hamiltonian is assumed invariant under the discrete translations, the eigenfunctions of the Hamiltonian are simultaneously eigenfunctions of $T(n)$.

The following properties of the translations allow us to determine the form of its eigenvalues:

$$\begin{aligned}
T(0) &= 1 \\
T(-n) &= T(n)^{-1} \\
T^\dagger(n)T(n) &= 1 \\
T(n)T(m) &= T(n+m) \\
T(n)T(m) &= T(m)T(n)
\end{aligned}$$

Label the eigenstates and eigenvalues of $T(n)$ as follows

$$T(n)\,|k\rangle = \lambda_n(k)\,|k\rangle$$

The above properties of T yield:

$$\begin{aligned}
\lambda_0(k) &= 1 \\
\lambda_{-n}(k) &= 1/(\lambda_n(k)) \\
|\lambda_n(k)|^2 &= 1 \\
\lambda_n(k)\lambda_m(k) &= \lambda_{n+m}(k) \\
\lambda_n(k)\lambda_m(k) &= \lambda_m(k)\lambda_n(k)
\end{aligned}$$

These relations imply that we can write $\lambda_n(k) = e^{-inka}$.

Denote the simultaneous eigenstates of H and $T(n)$ such that

$$\begin{aligned} H\,|E,k\rangle &= E\,|E,k\rangle \\ T(n)\,|E,k\rangle &= e^{-inka}\,|E,k\rangle \end{aligned}$$

Write the wave function in coordinate space as $\varphi_{E,k}(x) = \langle x|E,k\rangle \equiv \varphi(x)$.

$$\begin{aligned} \varphi(x) &= \langle x|E,k\rangle = \langle x'|T(n)|E,k\rangle \\ &= e^{-ik(na)}\langle x'|E,k\rangle = e^{-ik(na)}\varphi(x') \\ &= e^{-ik(x'-x)}\varphi(x') \end{aligned}$$

So the function

$$u(x) = \varphi(x)e^{-ikx} = \varphi(x')e^{-ikx'} = u(x') = u(x+na)$$

is periodic. Thus $\varphi(x) = u(x)e^{ikx}$, which means the eigenfunctions can be expressed as the product of a plane wave and a periodic function. This is Bloch's theorem.