Mathematical Physics

PHYS 3807

Across many types of physics, classical and quantum, we repeatedly encounter a limited set of fundamental equations. They are

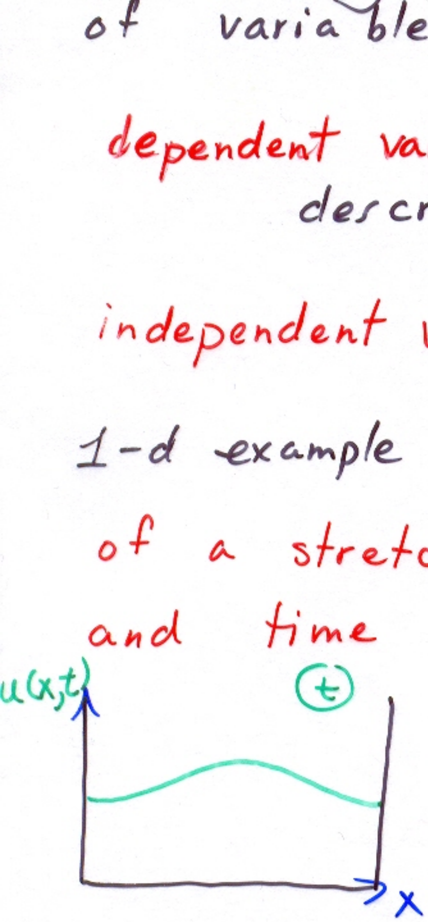
linear second order partial differential eqns. (pde)

In writing any diff. eq., we have two kinds of variable:

dependent variable: the "object" (field) that is described by the eq. It depends on the:

independent variables: spatial coordinates & time

1-d example to establish notation: The displacement of a stretched string as it depends on position and time is given by the 1-d wave eq:



$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2}$$

partial derivatives: pde

u: dependent variable

x,t: independent variables

$u(x,t)$

$\left[\text{cf. } \frac{dx(t)}{dt} = v\right]$ ODE

The diff. eq. is linear if only one factor of u or its derivatives occurs in each term.

No functions of u occur:

$$\frac{\partial^2 u(x,t)}{\partial x^2} + \underbrace{u^2(x,t)}_{\text{nonlinear}} = \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2}$$

Equations we see frequently include:

- inhomogeneous wave equation

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = -4\pi\rho$$

$$\frac{\partial^2}{\partial t^2}$$

Laplacian $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in Cartesian coords.

Arises in describing motion of string or membrane, with $\psi(\vec{r},t)$ representing displacement. Also describes other wave phenomena such as sound or light ($\vec{E}$, $\vec{B}$, potentials) waves or fluctuations in pressure or density. c is the speed of the wave propagation.

- heat conduction / diffusion equation

$$\nabla^2 \psi - \frac{1}{\kappa}\frac{\partial \psi}{\partial t} = -4\pi s \qquad \frac{\partial}{\partial t}$$

This describes the temperature in a system conducting heat or the diffusion of a substance. κ is the heat capacity or the diffusion constant in these respective situations.

Note that this eq is not invariant under $t \to -t$, unlike the wave eq. It describes irreversible processes.

- Helmholtz equation

$$\nabla^2 \psi + \lambda \psi = 0$$

This equation arises from the homogeneous versions of the wave eq and the heat conduction equations if we have

$$T(t) = e^{-i\omega t}$$

or

$$T(t) = e^{-\kappa\lambda t}$$

, respectively

and $\psi(\vec{r},t) = \phi(\vec{r})\,T(t)$

- Poisson's equation

$$\nabla^2 \psi = -4\pi\rho$$

$\leftarrow$ For $\rho = 0$, this reduces to Laplace's equation.

This arises in considering gravity or in electromagnetism. It also arises in the case of time independent wave or conduction/diffusion equations.

- Schrodinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

This arises in nonrelativistic quantum mechanics. Factoring out a time dependence $T(t) = e^{-iEt/\hbar}$ leads to the time independent S. eq:

$$\nabla^2\psi(\vec{r}) + \frac{2m}{\hbar^2}\left(E - V(\vec{r})\right)\psi(\vec{r}) = 0$$

For $V=0$, this is just the Helmholtz eq again

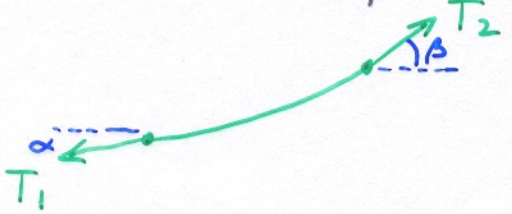
- Klein-Gordon equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \lambda^2\psi = 0$$

This arises in relativistic quantum mechanics.

→ • Wave equation : string under tension

Assume string cannot move longitudinally (horizontally) and that only horizontal force is that component of the tension.

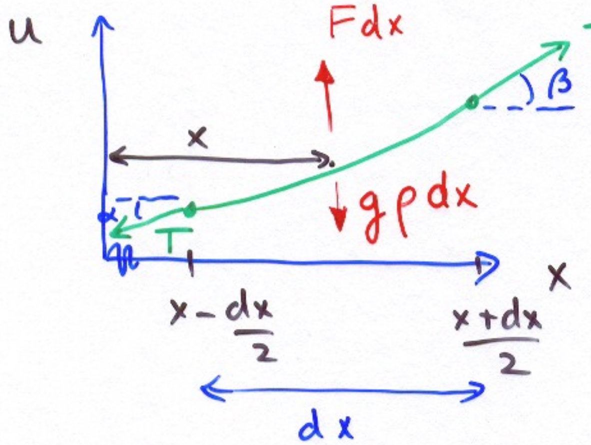


$$T_1 \cos\alpha = T_2 \cos\beta$$

Assume only small transverse (vertical) displacements $\longrightarrow$ α, β small

$\cos\alpha \sim 1 \qquad \cos\beta \sim 1$

$\rightarrow T_1 \sim T_2 \equiv T$



For element of length dx, Newton's 2nd law $\vec{F} = m\vec{a}$ has transverse component:

$$T(\sin\beta - \sin\alpha) + \underbrace{F\,dx}_{\text{external force}} - \overbrace{\rho\,dx}^{dm}\,g = (\rho\,dx)\frac{\partial^2 u}{\partial t^2}$$

For small angles α & β, $\sin\theta \sim \tan\theta =$ slope at the position

$$\sin\alpha \sim \tan\alpha = \left.\frac{\partial u}{\partial x}\right|_{(x-\frac{dx}{2})} \qquad \sin\beta \sim \tan\beta = \left.\frac{\partial u}{\partial x}\right|_{(x+\frac{dx}{2})}$$

$$\therefore \quad \frac{\sin\beta - \sin\alpha}{dx} \sim \frac{\left[\left.\frac{\partial u}{\partial x}\right|_{x+\frac{dx}{2}} - \left.\frac{\partial u}{\partial x}\right|_{x-\frac{dx}{2}} \right]}{dx} = \frac{\partial^2 u(x)}{\partial x^2}$$

$$T\frac{\partial^2 u}{\partial x^2} + \underbrace{F - \rho g} = \rho \frac{\partial^2 u}{\partial t^2}$$

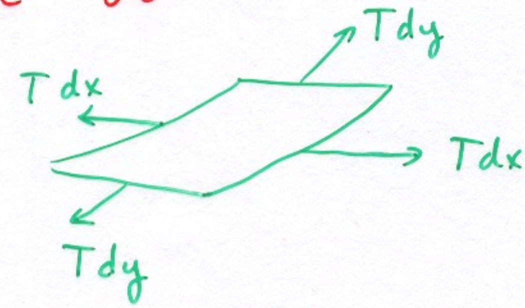
If no external forces and if weight is negligible relative to Tension:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{1-d wave eq.}$$

For a membrane:

$T =$ force / unit length

$\sigma =$ mass / unit area



Under the same assumptions, the vertical component of the force is:

$$T\,dx\left\{ \left.\frac{\partial u}{\partial y}\right|_{x,\,y+\frac{dy}{2}} - \left.\frac{\partial u}{\partial y}\right|_{x,\,y-\frac{dy}{2}} \right\} + T\,dy\left\{ \left.\frac{\partial u}{\partial x}\right|_{x+\frac{dx}{2},\,y} - \left.\frac{\partial u}{\partial x}\right|_{x-\frac{dx}{2},\,y} \right\}$$

$$= \sigma\, dx\, dy \frac{\partial^2 u}{\partial t^2}$$

$$\therefore \quad T\,dx\left\{ \frac{\partial^2 u}{\partial y^2}\,dy \right\} + T\,dy\left\{ \frac{\partial^2 u}{\partial x^2}\,dx \right\} = \sigma\, dx\, dy \frac{\partial^2 u}{\partial t^2}$$

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{again}$$

• heat conduction / diffusion equation

Considering the diffusion of a substance through a continuous medium <u>or</u> the conduction of heat leads to the same eq. — for the concentration (density) of the substance and the temperature, respectively; providing we begin with a linear law for the process.

Diffusion: $\vec{j} = -D\,\vec{\nabla}\rho$

Diffusion coefficient $\mathrm{cm^2/s}$

concentration or density $\frac{\#}{\mathrm{cm^3}}$ $\frac{\mathrm{gm}}{\mathrm{cm^3}}$

current density (amount [mass, #...] passing per unit time through unit area ⊥ to direction of flow)

$\frac{\text{"substance"}}{\mathrm{cm^2\cdot s}}$

Heat conduction: $\vec{q} = -k\,\vec{\nabla}T$

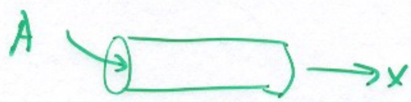
thermal conductivity $\frac{\mathrm{cal}}{\mathrm{cm\cdot s\cdot deg}}$

temperature deg

heat current density $\frac{\mathrm{cal}}{\mathrm{cm^2\cdot s}}$

Denote heat flowing per unit time in x-direction through cylinder of cross-sectional area A as:

$$Q = -kA\frac{\partial T}{\partial x}$$



$\frac{\text{heat}}{\text{time}}$ $\frac{\mathrm{cal}}{\mathrm{s}}$

So $\vec{j}$ and $\vec{q}$ represent <u>flux</u> - of substance or of heat

Can express the rate of change with time of the amount of a substance (or the amount of heat) in two ways:

$$\frac{dN}{dt} = \underbrace{\int_V d^3x \, \frac{\partial \rho}{\partial t}}_{\text{change of amount in a volume } V} = \underbrace{-\oint_S \vec{j} \cdot d\vec{A}}_{\text{amount flowing through bounding surface } S} = -\int_V \vec{\nabla} \cdot \vec{j} \, d^3x$$

by divergence thm (Gauss)

(This assumes no substance is created or destroyed in the volume.)

Since the above holds for any volume, we equate the integrands to find:

$$\vec{\nabla} \cdot \vec{j} = -\frac{\partial \rho}{\partial t}$$

(If there were a source of the substance in V

$$\frac{\partial \rho}{\partial t} \rightarrow \frac{\partial \rho}{\partial t} - 4\pi s \text{)}$$

Use $\vec{j} = -D \vec{\nabla} \rho$

$$\nabla^2 \rho - \frac{1}{D} \frac{\partial \rho}{\partial t} = 0 \qquad \left(= -\frac{4\pi s}{D} \right)$$

diffusion equation

In the case of heat conduction, we also have the relation between the amount of heat and the change in temperature:

$$Q = \int d^3x \; c \rho \, (\Delta T)$$

heat capacity (c): $\dfrac{\text{cal}}{\text{gm} \cdot \text{deg}}$

mass density (ρ): $\dfrac{\text{gm}}{\text{cm}^3}$

So
$$\frac{dQ}{dt} = \int_V d^3x \; c\rho \frac{\partial T}{\partial t} = -\oint_S \vec{q} \cdot d\vec{A} = -\int_V \vec{\nabla} \cdot \vec{q} \; d^3x$$

Equating the integrands, $\vec{\nabla} \cdot \vec{q} = -c\rho \dfrac{\partial T}{\partial t}$

use $\vec{q} = -k \vec{\nabla} T$

$$\nabla^2 T - \frac{c\rho}{k} \frac{\partial T}{\partial t} = 0$$

heat conduction equation

(if a source or sink is included $\dfrac{\partial T}{\partial t} \to \dfrac{\partial T}{\partial t} - 4\pi S$)

• Poisson's equation

Use electromagnetism as example.

Maxwell's equations:

① $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$

② $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

③ $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$

④ $\vec{\nabla}\cdot\vec{B} = 0$

For time independent phenomena, we have $\vec{\nabla}\times\vec{E}=0$, which implies that $\vec{E}$ can be written as the gradient of a scalar field, φ. $\boxed{\vec{\nabla}\times(\vec{\nabla}\varphi)=0}$

$\vec{E} = -\vec{\nabla}\varphi$ $\qquad\qquad$ $\underbrace{\vec{\nabla}\cdot\vec{E} = -\nabla^2\varphi = 4\pi\rho}_{\text{Poisson's eq.}}$

Poisson's equation reduces to Laplace's eq for $\rho=0$.

Parenthetically, it is useful to look at the magnetic field just to introduce the idea of gauge freedom.

$\vec{\nabla}\cdot\vec{B}=0$ implies $\vec{B}=\vec{\nabla}\times\vec{A}$ $\qquad$ $\boxed{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0}$

Put this into the time independent version of ② and use a vector identity.

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \frac{4\pi}{c} \vec{J}$$

We can eliminate the $\vec{\nabla}(\vec{\nabla} \cdot \vec{A})$ term and reduce this to the vector form of Poisson's eq. by noting that there is freedom in how we introduce the vector potential $\vec{A}$.

Since $\vec{\nabla} \times (\vec{\nabla}\lambda) = 0$, we can always add an arbitrary gradient to $\vec{A}$ and still satisfy Maxwell's eqns.

Choose $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$ such that $\vec{\nabla} \cdot \vec{A}' = 0$. That is, choose $\nabla^2 \lambda = -\vec{\nabla} \cdot \vec{A}$.

Then the unwanted term vanishes and we reduce to

$$\nabla^2 \vec{A} = -\frac{4\pi}{c} \vec{J}$$

All the equations mentioned are $\boxed{\text{linear.}}$

Consider the homogeneous case of a linear equation.

We see readily that linear homogeneous eqns. admit a $\boxed{\text{superposition of solutions.}}$

e.g. 2-d 2nd order

If ψ_1 and ψ_2 each satisfy

$$A\frac{\partial^2\psi}{\partial x^2} + B\frac{\partial^2\psi}{\partial y^2} + 2C\frac{\partial^2\psi}{\partial x\,\partial y} + D\frac{\partial\psi}{\partial x} + E\frac{\partial\psi}{\partial y} + F\psi = 0$$

then $(a\psi_1 + b\psi_2)$ also satisfies the equation. This result holds to any order.

If ϕ satisfies a nonhomogeneous equation and ψ_1 and ψ_2 satisfy the corresponding homo. eqn, them $(\alpha\phi + a\psi_1 + b\psi_2)$ satisfies the non homogeneous equation.

Boundary Conditions

So the set of types of eqns arising frequently includes:

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = -4\pi\rho \qquad \text{inhomo. wave}$$

$$\nabla^2 \psi - \frac{1}{K}\frac{\partial \psi}{\partial t} = -4\pi s \qquad \text{conduction / diffusion}$$

$$\nabla^2 \psi + \lambda \psi = 0 \qquad \text{Helmholtz}$$

$$\nabla^2 \psi = -4\pi\rho \qquad \text{Poisson} \quad (\text{Laplace } \rho = 0)$$

+ quantum mechanical wave equations.

What distinguishes the solns when the same form of eq arises under different circumstances?

It is boundary conditions that must ultimately be specified in order to explicitly solve for a particular physical situation. These are the conditions that give the behaviour of the soln on the boundary of its area of definition. In the case of time dependence, the $t=0$ behaviour gives the initial conditions.

Explicit boundary conditions:

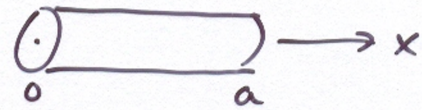
1. Dirichlet conditions: the value of the fn ψ is specified on the boundary

eg. stretched string fixed at $x=0$ and $x=a$

$$\psi(0,t) = \psi(a,t) = 0$$

2. (von) Neumann conditions: the value of the "normal derivative" is specified on the boundary

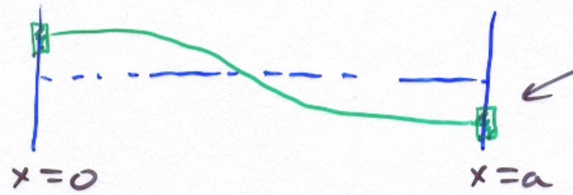
eg. $\vec{\nabla}\psi \cdot \hat{n} = 0$



insulated ends – no heat flow out of rod:

$$\left.\frac{\partial \psi}{\partial x}\right|_{x=0} = \left.\frac{\partial \psi}{\partial x}\right|_{x=a} = 0$$

or string with floating ends

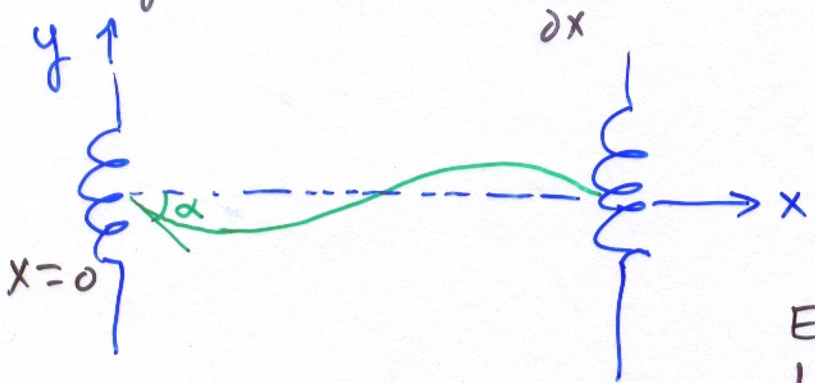


no force along the rods so string is horizontal at $x=0$ and $x=a$.

3. Cauchy conditions: both the fn and its derivative are specified on the boundary

4. Mixed – Robin's – intermediate conditions: a mix of Dirichlet and Neumann

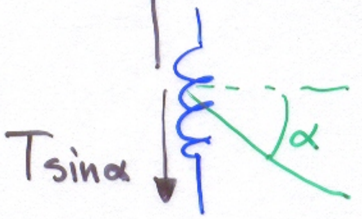
eg. $\psi(0,t) + \frac{\partial \psi(0,t)}{\partial x} = \psi(a,t) + \frac{\partial \psi(a,t)}{\partial x} = 0$



← $y=0$ equilibrium

Ends of string fixed to springs

Force from spring in y direction



$$F_y^{spring} = -k\,u(0,t)$$

spring constant

$$\tan\alpha = \text{slope} = \frac{\partial u(0,t)}{\partial x} \sim \sin\alpha$$

Balancing the two forces:

$$F_y^{spring} = -T_y$$

$$-k\,u(0,t) = T\frac{\partial u(0,t)}{\partial x}$$

(similarly at $x=a$)

5. Initial conditions: $\psi(x, t=0) = f(x)$

$$\frac{\partial \psi}{\partial x}(x, t=0) = g(x)$$

6. Periodicity

There are also many examples of implicit b.c.'s, based on our understanding of what is physically possible. eg. we may require normalisability of ψ. This is very important in q.m. where $|\psi|^2$ represents a probability density and we must maintain prob ≤ 1. Implies ψ must be finite at ∞ and must satisfy certain limit properties. (boundedness) eg. ψ may be required to be continuous and differentiable.

B.c.'s also influence a problem simply through their geometric character. May determine most approp choice of coords.

Orthogonal curvilinear coordinates

We can generally specify a point in 3-d space by three parameters:

(q_1, q_2, q_3) — (x, y, z) Cartesian

(ρ, ϕ, z) cylindrical

(r, θ, ϕ) spherical

We have some functional relationship between the Cartesian coordinates and any other set of coordinates.

$$x = x(q_1, q_2, q_3)$$

$$y = y(q_1, q_2, q_3)$$

$$z = z(q_1, q_2, q_3)$$

We assume this can be inverted:

$$q_i = q_i(x, y, z) \qquad i = 1, 2, 3$$

Each set of coordinates is defined by a set of intersecting surfaces defined by q_i = constant at each point. We call a coordinate system **curvilinear** if the orientation of these surfaces changes from point to point. It is **orthogonal curvilinear** if the surfaces are mutually perpendicular at each point.

Cartesian (x, y, z) 3 intersecting planes $x = x_1$, $y = y_1$, $z = z_1$

Cylindrical (ρ, φ, z) 2 planes and a cylinder $\rho = \sqrt{x^2+y^2} = \rho_1$, $\phi = \tan^{-1}(y/x) = \phi_1$, $z = z_1$

Spherical (r, θ, φ) sphere, cone, plane

$r = \sqrt{x^2+y^2+z^2} = r_1$ $\theta = \cos^{-1}(z/r) = \theta_1$ $\phi = \tan^{-1}(y/x) = \phi_1$

Note: coordinates do not necessarily represent lengths.

At any point, let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ be unit vectors orthogonal to the surfaces for q_1, q_2, q_3 respectively.

A point is specified by the radius vector from origin:

$$\vec{r} = x(q_i)\,\hat{\imath} + y(q_i)\,\hat{\jmath} + z(q_i)\,\hat{k}$$

An infinitesmal displacement $d\vec{r}$ is:

$$d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$$

where $dx = \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3$

$dy = \frac{\partial y}{\partial q_1} dq_1 + \cdots$

$dz = \cdots$

A displacement $\perp$ to one of the q_i = constant surfaces represents a displacement in the coordinate q_i, holding the other 2 constant.

eg. infinitesmal displacement $\perp$ to $q_1 = c$ surface yields:

$$d\vec{r}\Big|_{q_2,q_3} = (dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k})\Big|_{q_2,q_3}$$

$$= \left(\frac{\partial x}{\partial q_1}\hat{\imath} + \frac{\partial y}{\partial q_1}\hat{\jmath} + \frac{\partial z}{\partial q_1}\hat{k}\right) dq_1$$

Thus $\dfrac{\partial \vec{r}}{\partial q_1} = \dfrac{\partial x}{\partial q_1}\hat{\imath} + \dfrac{\partial y}{\partial q_1}\hat{\jmath} + \dfrac{\partial z}{\partial q_1}\hat{k}$

But $\dfrac{\partial \vec{r}}{\partial q_1}$ is a vector in the direction of $\hat{e}_1$ so

$$\hat{e}_1 = \frac{\dfrac{\partial \vec{r}}{\partial q_1}}{|\partial \vec{r}/\partial q_1|} = \frac{\left[\dfrac{\partial x}{\partial q_1}\hat{\imath} + \dfrac{\partial y}{\partial q_1}\hat{\jmath} + \dfrac{\partial z}{\partial q_1}\hat{k}\right]}{\sqrt{(\partial x/\partial q_1)^2 + (\partial y/\partial q_1)^2 + (\partial z/\partial q_1)^2}}$$

In general $\hat{e}_i = \dfrac{\partial \vec{r}/\partial q_i}{|\partial \vec{r}/\partial q_i|} \equiv \dfrac{1}{h_i}\dfrac{\partial \vec{r}}{\partial q_i}$

Note that $(d\vec{r})\Big|_{q_2, q_3} = h_1 dq_1 \hat{e}_1$ is the element of length dl, produced when only q_1 changes. In general $dl_i = h_i dq_i$

metric coefficients

$$d\vec{r} = h_1 dq_1 \hat{e}_1 + h_2 dq_2 \hat{e}_2 + h_3 dq_3 \hat{e}_3$$

An infinitesmal volume element is built as

$$d\tau = h_1 h_2 h_3 dq_1 dq_2 dq_3$$

Like wise area elements can be expressed.

Cylindrical	$h_\rho = 1$	$dl_\rho = d\rho$
	$h_\varphi = \rho$	$dl_\varphi = \rho d\varphi$
	$h_z = 1$	$dl_z = dz$
Spherical	$h_r = 1$	$dl_r = dr$
	$h_\theta = r$	$dl_\theta = r d\theta$
	$h_\varphi = r\sin\theta$	$dl_\varphi = r\sin\theta\, d\varphi$

The line element in an orthogonal curvilinear coordinate system is

$$ds = \sqrt{d\vec{r}\cdot d\vec{r}} = \sqrt{h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2}$$

All the differential operators follow nicely.

Express gradient using its basic definition:

$$df = \vec{\nabla} f \cdot d\vec{r} = \frac{\partial f}{\partial q_1} dq_1 + \frac{\partial f}{\partial q_2} dq_2 + \frac{\partial f}{\partial q_3} dq_3$$

where f is a function $f(q_1, q_2, q_3)$

$$= (\vec{\nabla} f)_1 \underbrace{h_1 dq_1}_{(d\vec{r})_1} + (\vec{\nabla} f)_2 \underbrace{h_2 dq_2}_{(d\vec{r})_2} + (\vec{\nabla} f)_3 \underbrace{h_3 dq_3}_{(d\vec{r})_3}$$

This implies $(\vec{\nabla} f)_i = \dfrac{1}{h_i} \dfrac{\partial f}{\partial q_i}$

$$\boxed{\vec{\nabla} f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\hat{e}_3}$$

General expressions for divergence and curl in curvilinear coordinates are found using their fundamental definitions via the divergence thm and Stoke's thm, respectively.

$$(\vec{\nabla} \times \vec{A})_n = \lim_{S \to 0} \oint \vec{A} \cdot d\vec{l}$$

(component of curl $\perp$ to surface S)

$$\vec{\nabla} \cdot \vec{A} = \lim_{V \to 0} \frac{1}{V} \oint_S \vec{A} \cdot \hat{n} \underbrace{da}_{\text{area element}}$$

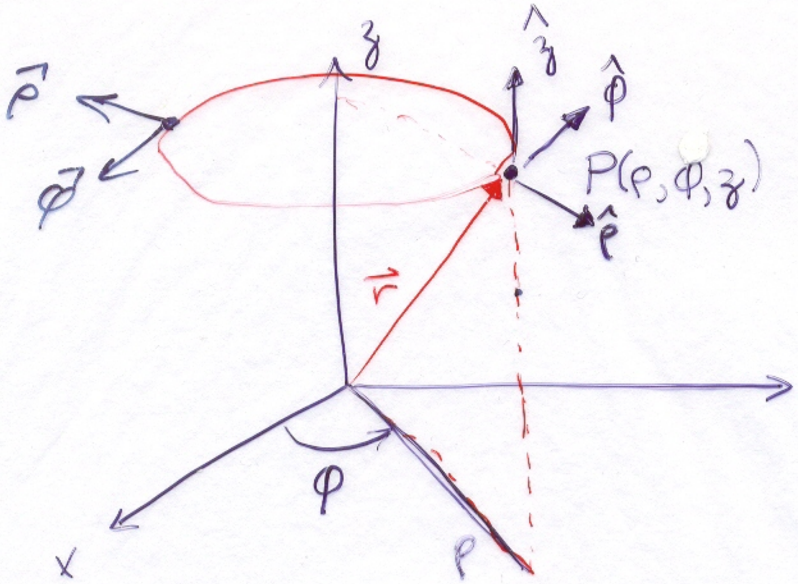
Results:

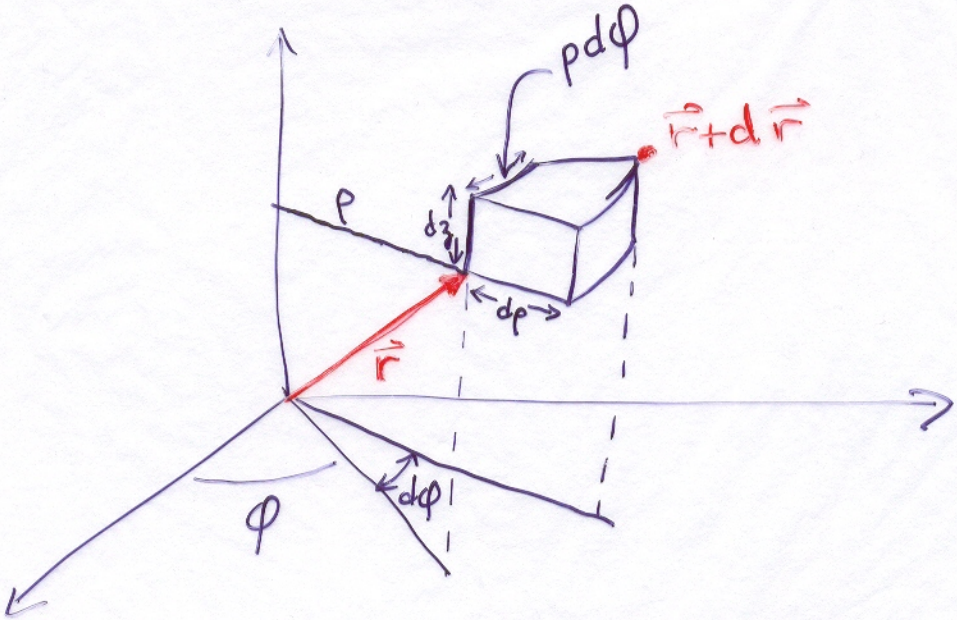
$$\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 A_1) + \frac{\partial}{\partial q_2}(h_1 h_3 A_2) + \frac{\partial}{\partial q_3}(h_1 h_2 A_3)\right]$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

The Laplacian $\nabla^2\phi = \vec{\nabla}\cdot\vec{\nabla}\phi$ is

$$\nabla^2\phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial\phi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial\phi}{\partial q_3}\right)\right]$$





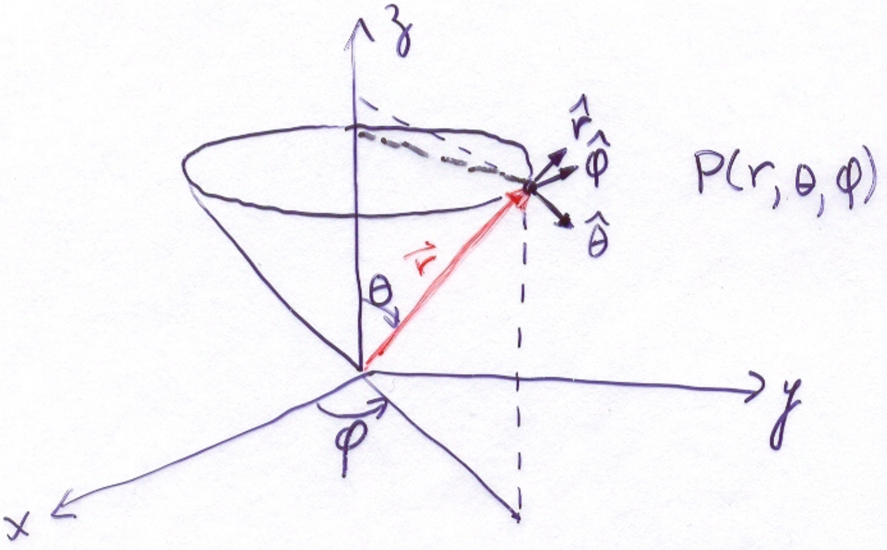
$$d\vec{r} \sim dz\,\hat{z} + d\rho\,\hat{\rho} + \rho\, d\phi\,\hat{\phi}$$

Generalized coords.

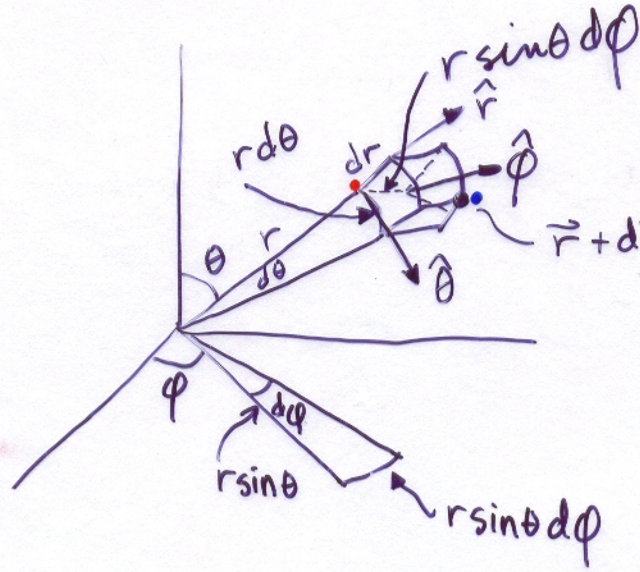
$q_1 = \rho \qquad dl_1 = d\rho \equiv h_1\, dq_1$

$q_2 = \phi \qquad dl_2 = \rho\, d\phi \equiv h_2\, dq_2$

$q_3 = z \qquad dl_3 = dz \equiv h_3\, dq_3$



$$\vec{r} = r\,\hat{r}$$



$$d\vec{r} \sim dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\varphi\,\hat{\varphi}$$

$q_1 = r$	$dl_1 = dr$
$q_2 = \theta$	$dl_2 = r\,d\theta$
$q_3 = \varphi$	$dl_3 = r\sin\theta\,d\varphi$

Separation of variables

Reduce pde to a set of ordinary differential eqs.

Choice of coordinates determined by geometry (b.c.s)

Resulting ODEs are essentially eigenvalue problems for a set of differential operators. Solutions are expressed as an expansion in the eigenfunctions corresponding to eigenvalues. Eigenvalues are only explicitly determined once the boundary conditions are imposed.

- time dependence

Both wave eq + heat conduction/diffusion eq have a time dependence. Consider their 3-d homogeneous versions and assume the time dependence of the solutions factorizes.

Wave eq $\qquad \nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$

$$\psi(\vec{r}, t) = u(\vec{r})\, T(t)$$

$$T(t)\, \nabla^2 u(\vec{r}) - \frac{1}{c^2} u(\vec{r}) \frac{d^2 T}{dt^2} = 0$$

Divide by ψ:

$$\frac{\nabla^2 u(\vec{r})}{u(\vec{r})} = \frac{1}{c^2 T}\frac{d^2 T}{dt^2}$$

LHS - spatial coords only

RHS - time only

Each side must be equal to a (separation) constant

$$\frac{d^2T}{dt^2} = -\lambda c^2 T$$

$$\nabla^2 u = -\lambda u$$ Helmholtz equation (Laplace for $\lambda = 0$)

The appropriate solution for the eqn in time depends on the sign of λ, which is normally determined by the spatial boundary conditions.

$\lambda > 0$ $\quad T_\lambda(t) = A \cos\sqrt{\lambda}\, ct + B \sin\sqrt{\lambda}\, ct$

(or, equivalently, $T_\lambda(t) = A e^{i\sqrt{\lambda}ct} + B e^{-i\sqrt{\lambda}ct}$)

$\lambda < 0$ $\quad T_\lambda(t) = A' e^{\sqrt{-\lambda}ct} + B' e^{-\sqrt{-\lambda}ct}$

$\lambda = 0$ $\quad T_\lambda(t) = \tilde{A} ct + \tilde{B}$

The homogeneous heat conduction / diffusion eq, $\nabla^2 \psi - \frac{1}{\kappa}\frac{\partial \psi}{\partial t} = 0$, reduces to:

$$\nabla^2 u + \lambda u = 0$$ Helmholtz again

$$\frac{dT}{dt} = -\lambda \kappa T$$

General solutions for the time dependence are:

$\lambda > 0 \qquad T_\lambda(t) = A e^{-\lambda K t}$

$\lambda < 0 \qquad T_\lambda(t) = A' e^{\lambda K t}$

$\lambda = 0 \qquad T_\lambda(t) = \tilde{A}$

From wave eq, we got a 2nd order ODE for time dependence. So general soln is always a superposition of 2 linearly independent solns. For cond/diff, we have a first order ODE so there is just one form for the solution in each case.

Let's look at a very simple 1-d example to see the eigenvalue problem concept. Follow this by looking at the 3-d spatial dependence to reinforce the separation of variables method and to practice with the various coordinate systems.

Simple example: Stretched string of length L

The displacement is described by 1-d wave eq.

$$\frac{d^2 u(x)}{dx^2} = -\lambda u(x)$$

$\lambda > 0 \qquad u(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$

$\lambda < 0 \qquad u(x) = A' e^{\sqrt{-\lambda}\,x} + B' e^{-\sqrt{-\lambda}\,x} \qquad (t)$

$\lambda = 0 \qquad u(x) = \tilde{A}x + \tilde{B}$

Boundary conditions dictate the choice among these general possibilities.

Consider the string to be fixed at its ends.

$$u(0,t) = u(L,t) = 0$$

* Known as homogeneous b.c.'s

Assume an arbitrary initial state

$$u(x,0) = u_0(x)$$

$$\frac{\partial u(x,0)}{\partial t} \equiv \left.\frac{\partial u}{\partial t}\right|_{t=0} = v_0(x)$$

The spatial b.c. can only be satisfied by the $\lambda > 0$ solutions and only if $A = 0$.

homogeneous b.c. at one end $(x=0)$ of the range in x determines the eigenfunction.

To satisfy both spatial b.c.'s must have $\sqrt{\lambda_n}\,L = n\pi$ $\quad n = 1, 2, 3 \ldots$

homo b.c. at other end of range in x $(x=L)$ determines the eigenvalues

So we have an infinite set of solutions.

$$u_n(x) = B_n \sin\sqrt{\lambda_n}\, x = B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad n = 1, 2, 3 \ldots$$

The $u_n(x)$ are eigenfunctions of the operator $\frac{d^2}{dx^2}$ and the λ_n are its eigonvalues.

Since the λ_n must be positive, this determines the time dependence as:

$$T_n(t) = C_n \cos \frac{n\pi ct}{L} + D_n \sin \frac{n\pi ct}{L}$$

So we can write the solutions, of which there are an infinite #, as

$$\psi_n(x,t) = u_n(x)\, T_n(t)$$

$$= \left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right) \sin \frac{n\pi x}{L}$$

These are known as the characteristic or normal modes of vibration of the string, each of eigenfrequency $\omega_n = \frac{n\pi c}{L}$.

The constant coefficients, A_n and B_n, must be determined by the initial conditions.

Every linear combination of the solutions is also a soln. In order to reproduce the initial conditions, we superpose solns as:

$$\psi(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right) \sin \frac{n\pi x}{L}$$

We must have:

$$\psi(x,0) = u_0(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$$

$$\left.\frac{\partial \psi}{\partial t}\right|_{t=0} = v_0(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin\frac{n\pi x}{L}$$

Assuming continuous fns, these are Fourier sine series converging to $\psi(x,t)$, $u_0(x)$, $v_0(x)$, with the coefficients determined as:

$$A_m = \frac{2}{L}\int_0^L dx \; u_0(x) \sin \frac{m\pi x}{L}$$

$$B_m = \frac{2}{m\pi c}\int_0^L dx \; v_0(x) \sin \frac{m\pi x}{L}$$

Have used the orthogonality property of the trigonometric functions

$$\int_0^L dx \; \sin\frac{n\pi x}{L} \sin \frac{m\pi x}{L} = \frac{L}{2}\delta_{nm}$$

To summarize, the solution $\psi(x,t)$ can be expressed as an eigenfunction expansion

$$\psi(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{L}$$

Each term in this series satisfies the spatial boundary conditions. Putting this back into the wave eq and interchanging the order of differentiation and summation:

$$\sum_{n=1}^{\infty} b_n(t) \frac{d^2}{dx^2}\left(\sin \frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n(t)\left(\frac{-n^2\pi^2}{L^2}\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{m=1}^{\infty} \frac{1}{c^2}\left(\frac{d^2 b_m(t)}{dt^2}\right) \sin \frac{m\pi x}{L}$$

From orthogonality of the sine fns, pull out the time dependence:

$$\frac{d^2 b_n(t)}{dt^2} + \frac{n^2\pi^2 c^2}{L^2} b_n(t) = 0$$

This has solutions

$$b_n(t) = A_n \cos \frac{n\pi c t}{L} + B_n \sin \frac{n\pi c t}{L}$$

with A_n and B_n determined by initial conditions as above.

Spatial Dependence : 3-d Helmholtz eq.
(Laplace)

For both wave eq and cond/diff eq, the spatial dependence reduces to Helmholtz eq. Consider the reduction of that pde by separation of the spatial variables in the 3 coordinate systems.

$$\nabla^2 \psi(\vec{r}) + \lambda \psi(\vec{r}) = 0$$

Cartesian $\quad \psi(\vec{r}) = X(x)\, Y(y)\, Z(z)$

$$YZ \frac{d^2X}{dx^2} + XZ \frac{d^2Y}{dy^2} + XY \frac{d^2Z}{dz^2} + \lambda XYZ = 0$$

Divide by XYZ:

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{x \text{ only}} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{y \text{ only}} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{z \text{ only}} = -\lambda$$

$$\frac{d^2X}{dx^2} = -\lambda_1 X \qquad \frac{d^2Y}{dy^2} = -\lambda_2 Y \qquad \frac{d^2Z}{dz^2} = -\lambda_3 Z$$

$$\lambda_1 + \lambda_2 + \lambda_3 = \lambda$$

All same form as time dependence eq. Specific solutions depend on the sign of the separation constants, as determined by boundary conditions.

Specific ~~ex~~ ample to illustrate the general method:

Find the electrostatic potential inside a rectangular metal box of dimensions $l \times w \times h$, with all sides grounded ($V=0$) except the top, which is held at constant potential V.

Laplace's eq: $\nabla^2 \psi = 0$

Geometry:
$$0 \le x \le l$$
$$0 \le y \le w$$
$$0 \le z \le h$$

Boundary conditions: $\psi(x, y, h) = V$

$$\psi(0, y, z) = \psi(l, y, z) = \psi(x, 0, z) = \psi(x, w, z) = \psi(x, y, 0)$$

1) Choose appropriate coordinate system $\longrightarrow$ Cartesian

2) Separate the pde into a set of ODE's.

Laplace is just Helmholtz with $\lambda = 0$.

$$\frac{d^2 X}{dx^2} = -\lambda_1 X \qquad \frac{d^2 Y}{dy^2} = -\lambda_2 Y \qquad \frac{d^2 Z}{dz^2} = -\lambda_3 Z$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 0$$

Depending on the sign of the λ_i's, the solutions are of the forms (†) on page 24.

3) Consider the coordinate(s) for which the b.c.'s are homogeneous at both ends of its range. Impose those b.c.'s. Repeat for any other coordinate with two homogeneous b.c.'s.

In this case, both x and y have two homogeneous b.c.'s.

Imposing $X(x=0)=0$ and $Y(y=0)=0$ implies they each take the oscillatory form, in particular the sine form.

Thus λ_1 and λ_2 must both be positive. This implies λ_3 is negative since $\lambda_1+\lambda_2+\lambda_3=0$.

$$X(x) = A_x \sin\sqrt{\lambda_1}\, x$$

$$Y(y) = A_y \sin\sqrt{\lambda_2}\, y$$

Imposing: $X(l)=0$ implies $\sqrt{\lambda_{1n}} = \dfrac{n\pi}{l}$ $\quad n=1,2,\ldots$

$Y(w)=0$ implies $\sqrt{\lambda_{2m}} = \dfrac{m\pi}{w}$ $\quad m=1,2,\ldots$

$$\lambda_3 = -\left[\left(\frac{n\pi}{l}\right)^2 + \left(\frac{m\pi}{w}\right)^2\right]$$

4) Solve for the remaining function using the final homogeneous b.c.

Since $\lambda_3 < 0$, $Z(z) = A e^{\sqrt{-\lambda_3}\, z} + B e^{-\sqrt{-\lambda_3}\, z}$

$Z(0) = 0$ implies $A + B = 0$ so $Z(z)$ takes a hyperbolic sine (sinh) form.

Thus, general form of our factored eigenfunction:

$$\psi_{nm}(x,y,z) = A_{nm} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi y}{w}\right) \sinh\left(\sqrt{\left(\frac{n\pi}{l}\right)^2 + \left(\frac{m\pi}{w}\right)^2}\, z\right)$$

5) Form the general solution as a linear superposition:

$$\psi = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin\frac{n\pi x}{l} \sin\frac{m\pi y}{w} \sinh\left(\sqrt{\left(\frac{n\pi}{l}\right)^2 + \left(\frac{m\pi}{w}\right)^2}\, z\right)$$

6) Impose the final boundary condition.

$$\psi(x,y,h) = V$$

$$V = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin\frac{n\pi x}{l} \sin\frac{m\pi y}{w} \sinh\left(\sqrt{\left(\frac{n\pi}{l}\right)^2 + \left(\frac{m\pi}{w}\right)^2}\, h\right)$$

Use the orthogonality of the sine fns to find A_{mn}.

Multiply by $\sin\frac{p\pi x}{l}$ and $\sin\frac{q\pi y}{w}$ and integrate over the range of x $(0 \to l)$ and of y $(0 \to w)$.

$$\int_0^w dy \int_0^l dx \; V \sin\frac{p\pi x}{l} \sin\frac{q\pi y}{w} =$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \underbrace{\int_0^l dx \sin\frac{p\pi x}{l} \sin\frac{n\pi x}{l}}_{\frac{l}{2}\delta_{pn}} \underbrace{\int_0^w dy \sin\frac{q\pi y}{w} \sin\frac{m\pi y}{w}}_{\frac{w}{2}\delta_{qm}} \cdot \sinh(\;)$$

$$V\left[\frac{w}{q\pi}\left[1-(-1)^q\right]\right]\left[\frac{l}{p\pi}\left[1-(-1)^p\right]\right] = A_{pq}\left(\frac{l}{2}\right)\left(\frac{w}{2}\right)\sinh(\sqrt{\;}\,h)$$

$$A_{pq} = \frac{4}{lw} V \frac{wl}{qp\pi^2} \frac{\left[1-(-1)^q\right]\left[1-(-1)^p\right]}{\sinh\left(\sqrt{\left(\frac{p\pi}{l}\right)^2+\left(\frac{q\pi}{w}\right)^2}\; h\right)}$$

This is nonvanishing only for p, q odd in which case each square bracket factor is 2.

$$\psi(x,y,z) =$$

$$\sum_{n=1,3,5\ldots} \sum_{m=1,3,5\ldots} \frac{16V}{\pi^2} \frac{1}{nm} \sin\frac{n\pi x}{l} \sin\frac{m\pi y}{w} \frac{\sinh\left(\sqrt{\left(\frac{n\pi}{l}\right)^2+\left(\frac{m\pi}{w}\right)^2}\; z\right)}{\sinh\left(\sqrt{\left(\frac{n\pi}{l}\right)^2+\left(\frac{m\pi}{w}\right)^2}\; h\right)}$$

Back to our set of pde's in the various coordinate systems. The homogeneous versions are:

Laplace $\nabla^2 \psi = 0$

Diff/cond $\nabla^2 \psi - \frac{1}{K}\frac{\partial \psi}{\partial t} = 0$

Wave $\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$

$\left.\right\} \psi(\vec{r},t) = u(\vec{r})T(t)$

The time dependence for diff $\frac{dT}{dt} = -\lambda K T$ pg 24

wave $\frac{d^2T}{dt^2} = -\lambda c^2 T$ pg 23

$\boxed{\text{Conditions on } t\text{-dependence}}$

In the diff eq. case, let's simply assume that $T(t)$ is bounded over $0 \le t < \infty$: $|T(\infty)| < \infty$.
For the diffusion eq, this implies

$$T_\lambda(t) = A_\lambda e^{-\lambda K t} \qquad \text{with } \boxed{\lambda > 0}$$

Define $\lambda \equiv k^2$ $\quad T_k(t) = A_k e^{-k^2 K t}$ and

the eq for $u(\vec{r})$ is $\nabla^2 u + k^2 u = 0$

For the wave equation, impose boundedness on the interval $-\infty < t < \infty$, $|T(\pm\infty)| < \infty$. Thus

$$T_\lambda(t) \underset{\lambda \equiv k^2}{=} T_k(t) = A_k e^{ikct} + B_k e^{-ikct}$$

Helmholtz eq $\nabla^2 u + k^2 u$ arises again.

$k^2 > 0$

As before it reduces to 3 ODE's:

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda_1 \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = -\lambda_2 \qquad \frac{1}{Z}\frac{d^2Z}{dz^2} = -\lambda_3$$

where $\lambda_1 + \lambda_2 + \lambda_3 = k^2$

Conditions on spatial dependence

Assume infinite spatial medium and boundedness for the solution for all x, y, z. This requires that each $\lambda_i \geq 0$.

$$\lambda_1 \equiv k_1^2 \qquad \lambda_2 \equiv k_2^2 \qquad \lambda_3 \equiv k_3^2 \qquad -\infty < k_1, k_2, k_3 < \infty$$

$$X(x) \propto e^{ik_1 x} \qquad Y(y) \propto e^{ik_2 y} \qquad Z(z) \propto e^{ik_3 z}$$

$$u(\vec{r}) = A_k\, e^{i\vec{k}\cdot\vec{r}} \qquad |\vec{k}| = k$$

plane wave soln.

Cylindrical Coordinates (ρ, ϕ, z)

Using the same assumptions for boundedness of the time dependent factor, which implies that the separation constant is positive, let's look at Helmholtz eqn in cylindrical coordinates:

Note - change of notation

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} + \lambda^2 u = 0$$

This can also be written as

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}+\lambda^2 u=0$$

Assume we can separate the variables

$$u(\rho,\varphi,z)=R(\rho)\,\Phi(\varphi)\,Z(z)$$

Divide by u

$$\underbrace{\frac{1}{R\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)}_{\frac{1}{R}\left[\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}\right]}+\frac{1}{\rho^2}\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}+\lambda^2=0$$

depends on ρ,φ only

depends on z only

Thus, each must be a constant.

Denote: $$\frac{1}{Z}\frac{d^2Z}{dz^2}=k^2-\lambda^2$$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}=-m^2$$

* Note: No assumption made here about the sign of k^2.

Then $$\frac{1}{R}\left[\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}\right]-\frac{m^2}{\rho^2}+k^2=0$$

This is our new set of 3 ODE's for cases of cylindrical symmetry.

The equation in φ we've seen before and it's simple to solve. We'll write the solution from the perspective that we typically require it to be single valued in the sense that

$$\Phi(\varphi + 2\pi) = \Phi(\varphi)$$

This condition can only be satisfied by the oscillatory form of the solns and, further, imposes a restriction on the separation constant m.

$$\Phi(\varphi) = A e^{im\varphi} + B e^{-im\varphi}$$

$$= A' \cos m\varphi + B' \sin m\varphi$$

where m is an integer $\quad m = 0, 1, 2, \cdots$

The equation in z has the same simple form.

$$Z(z) = C e^{\sqrt{k^2-\lambda^2}\, z} + D e^{-\sqrt{k^2-\lambda^2}\, z}$$

This will either take the (oscillatory) trigonometric form or the hyberbolic trig form depending on the sign of the argument of the square root.

The radial equation

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \frac{m^2}{\rho^2}R + k^2 R = 0$$

is conventionally rewritten in terms of a new scaled variable $x \equiv k\rho$. It becomes

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{m^2}{x^2}\right) R = 0$$

This is Bessel's equation of order m.

Its two linearly independent solutions are the Bessel (J_m) and Neumann (N_m) functions.

$$R(k\rho) = A\, J_m(k\rho) + B\, N_m(k\rho)$$

The Hankel functions of the first and second Kind are convenient combinations:

$$H_m^{(1)}(x) = J_m(x) + i N_m(x)$$

$$H_m^{(2)}(x) = J_m(x) - i N_m(x)$$

When the sign of the separation constant k^2 is such that the Z solns are oscillatory, we get the modified Bessel eq, whose solns have imaginary argument & are conventionally written in terms of the modified Bessel fns with real argument $I_m(x)$ and $K_m(x)$.

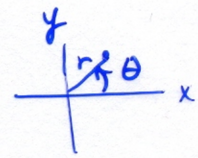
Will look at this in more detail later. Before moving on, note one special case:

Say we have Laplace's eqn rather than Helmholtz

$$\nabla^2 u = 0 \qquad\qquad \nabla^2 u + \lambda^2 u = 0$$

and say there is no z dependence in the problem.

$$\frac{1}{R}\left[\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho}\right] + \frac{1}{\rho^2}\underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}}_{= -m^2} = 0$$



So this is just a 2-d problem in polar coordinates. Let's write ρ as r and ϕ as θ in this case. The radial eqn is now:

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{m^2}{r^2}R = 0$$

For $m=0$, the general soln to this equation is

$$R(r) = G_0 + H_0 \ln r$$

For $m \neq 0$, we have

$$R(r) = G_m r^m + H_m r^{-m}$$

Thus, the general superposition of solutions is

$$u(r,\theta) = A_0 + B_0 \ln r + \sum_{m=1}^{\infty} (A_m r^m + B_m r^{-m})(C_m \cos m\theta + D_m \sin m\theta)$$

(a Fourier series in θ)

Spherical Coordinates (r, θ, φ)

(note $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)$)

Helmholtz's eqn is

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 \psi}{\partial\varphi^2}\right] + k^2\psi = 0$$

Go through the usual drill. For now, put $\psi(r,\theta,\varphi) = R(r)\, Y(\theta,\varphi)$ and divide by RY.

$$\frac{1}{Rr}\frac{d^2(rR)}{dr^2} + \frac{1}{Y r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right] + k^2 = 0$$

Separating the angular dependence, we write

$$\frac{1}{Y\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right] = -\lambda \qquad **$$

which leaves the radial equation

$$\frac{1}{Rr}\frac{d^2(rR)}{dr^2} + k^2 - \frac{\lambda}{r^2} = 0$$

For $k^2 \neq 0$, we can put this radial equation into the form of Bessel's equation as follows.

Define a new variable $x \equiv kr$

$$\frac{1}{Rx} k^2 \underbrace{\frac{d^2}{dx^2}(xR)}_{\frac{d}{dx}\left(R + x\frac{dR}{dx}\right)} + k^2\left(1 - \frac{\lambda}{x^2}\right) = 0$$

$$\frac{1}{Rx}\left(2\frac{dR}{dx} + x\frac{d^2R}{dx^2}\right) + \left(1 - \frac{\lambda}{x^2}\right) = 0$$

$$\frac{d^2R}{dx^2} + \frac{2}{x}\frac{dR}{dx} + \left(1 - \frac{\lambda}{x^2}\right)R = 0 \qquad *$$

Now scale the function R to rewrite the equation in terms of a new function y.

$$R = \frac{1}{\sqrt{x}}\, y$$

$$\frac{dR}{dx} = -\frac{1}{2}x^{-3/2}\, y + x^{-1/2}\frac{dy}{dx}$$

$$\frac{d^2R}{dx^2} = \frac{3}{4}x^{-5/2}\, y - x^{-3/2}\frac{dy}{dx} + x^{-1/2}\frac{d^2y}{dx^2}$$

Plugging into * and simplifying:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left[1 - \frac{\left(\lambda + \frac{1}{4}\right)}{x^2}\right]y = 0$$

This is Bessel's eqn with m^2 replaced by $\left(\lambda + \frac{1}{4}\right)$.
($R = \frac{y}{\sqrt{x}} \sim \frac{J}{\sqrt{x}} \sim j$ spherical Bessel fns)

(figure after pg 43)

Laplace's eqn.

For the case that $k^2 = 0$, the radial equation reduces to

$$\frac{1}{r}\frac{d^2}{dr^2}(rR) - \frac{\lambda}{r^2} R = 0$$

Assuming a power law solution, $R \propto r^\alpha$, yields an equation for the power α.

$$r^{-1}(\alpha+1)\alpha r^{\alpha-1} - \lambda r^{\alpha-2} = 0$$

$$\alpha(\alpha+1) - \lambda = 0$$

$$\alpha = \frac{1}{2}\left\{-1 \pm \sqrt{1+4\lambda}\right\}$$ Two roots α_1, α_2.

So for Laplace's equation, in spherical coordinates, the radial factor of the solution is

$$R = A r^{\alpha_1} + B r^{\alpha_2}$$

Now return to the angular dependence of Helmholtz equation. We separate the dependence of the two angles by writing $Y(\theta,\varphi) = P(\theta)\,\Phi(\varphi)$. Eq ** on page 40 becomes:

$$\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{1}{\sin^2\theta}\underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}}_{\equiv -m^2} = -\lambda$$

Take m^2 non negative so that the periodic form of the soln Φ is chosen. Then assume single valued: $\Phi(\varphi+2\pi) = \Phi(\varphi)$

$$\Phi(\varphi) = A_m e^{im\varphi} \qquad m = 0, \pm 1, \pm 2, \ldots$$

This leaves

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) + \left(\lambda - \frac{m^2}{\sin^2\theta}\right) P = 0$$

This equation looks simpler if we do a change of variables to $x = \cos\theta$:

$$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) + \left(\lambda - \frac{m^2}{(1-x^2)}\right) P = 0$$

This is the Associated Legendre equation, which we will study later. For $m=0$, the equation is called the Legendre equation.

Solutions Legendre polynomials and associated L poly.

plus L fns of 2nd Kind

General properties of linear second order ODEs and their series solutions

We study the general class of equations written below in "standard form".

$$y''(x) + p(x)\,y'(x) + q(x)\,y(x) = r(x)$$

$r(x) = 0 \longrightarrow$ homogeneou

$$y' \equiv \frac{dy}{dx}$$

Consider homogeneous equation. Introduce the idea of singular points of the equation.

A point $x = x_0$ is called an ordinary or regular point of the eq. if $p(x_0)$ and $q(x_0)$ are finite (namely, not infinite).

If either $p(x)$ or $q(x)$ diverges at $x = x_0$, then x_0 is called a singular point.

We further subdivide the notion of singular point

Regular singular point: either $p(x)$ or $q(x)$ may diverge at $x = x_0$ but $(x-x_0)\,p(x)$ and $(x-x_0)^2 q(x)$ remain finite at $x = x_0$.

$\longrightarrow$ namely, simple pole in $p(x)$, double pole in $q(x)$

Irregular singular pole: either $(x-x_0)p(x)$ or $(x-x_0)^2 q(x)$ diverges at $x=x_0$.

We mostly work with equations such that $p(x)$ and $q(x)$ are analytic functions in some region of x except for isolated poles.

A fn is analytic at a point x_0 if it can be expanded in a Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x)\Big|_{x=x_0} (x-x_0)^n$$

that converges in the neighborhood of x_0.

(We should also be able to consider the behaviour at $x=\infty$. To do so, make the change of variable $z = x^{-1}$

Then the general eqn becomes, after a little algebra,

$$\frac{d^2y}{dz^2} + \underbrace{\frac{(2z-p)}{z^2}}_{\equiv \tilde{p}} \frac{dy}{dz} + \underbrace{\frac{q}{z^4}}_{\equiv \tilde{q}} y = 0$$

Here everything is considered as a function of $z = \frac{1}{x}$.

$y(1/z)$, $p(1/z)$, $q(1/z)$

So we see that $x=\infty$ is an ordinary point if $\tilde{p} = \frac{(2z-p)}{z^2}$ and $\tilde{q} = \frac{q}{z^4}$ are both finite at $z=0$. The point $x=\infty$ is a regular singular pt if $\tilde{p}$ or $\tilde{q}$ diverges but $z\tilde{p}$ and $z^2\tilde{q}$ are finite at $z=0$.
If $z\tilde{p}$ or $z^2\tilde{q}$ diverge at $z=0$, then $x=\infty$ is an irregular singular point.

Check out the singularity structure of a couple of the eqns we have encountered.

① Using cylindrical coords, we found the radial dependence described by Bessel's eqn:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{m^2}{x^2}\right) y = 0$$

$$p(x) = \frac{1}{x} \qquad q(x) = 1 - \frac{m^2}{x^2}$$

Both p and q diverge at $x=0$ but $xp(x)=1$ and $x^2 q(x) = x^2 - m^2$ are finite at $x=0$.
Thus $x=0$ is a regular singular point for Bessel's eq.

Checking out the point at ∞,

$$p(x) = \frac{1}{x} \longrightarrow p(1/z) = z$$

$$q(x) = 1 - \frac{m^2}{x^2} \longrightarrow q(1/z) = 1 - m^2 z^2$$

$$\therefore \quad \tilde{p} = \frac{(2z - z)}{z^2} = \frac{2}{z}$$

$$\tilde{q} = \frac{(1 - m^2 z^2)}{z^4}$$

The $\frac{1}{z^4}$ pole at $z=0$ in $\tilde{q}$ shows that the point $x=\infty$ is an irregular singular point.

② In spherical coordinates, we had the associated Legendre equation for the $x = \cos\theta$ dependence:

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{dy}{dx}\right) + \left(\lambda - \frac{m^2}{(1-x^2)}\right) y = 0$$

In standard form,

$$y'' - \frac{2x}{(1-x^2)} y' + \left[\frac{\lambda}{1-x^2} - \frac{m^2}{(1-x^2)^2}\right] y = 0$$

$p(x) = \dfrac{-2x}{(1-x)(1+x)}$ diverges at $x=+1$ and $x=-1$ but $(x-1)\,p(x)$ is finite at $x=+1$ and $(x+1)\,p(x)$ is finite at $x=-1$.

Similarly, $q(x) = \dfrac{\lambda}{1-x^2} - \dfrac{m^2}{(1-x^2)^2}$ is OK when multiplied by $(x-x_0)^2$.

Thus $x=+1$ and $x=-1$ are regular singular points for the associated Legendre eqn.

Same result for $m=0$ case: Legendre eqn.

Back to our equation:

$$\underset{\substack{\uparrow\\ \text{operator}}}{L}\, y = y''(x) + p(x)\,y'(x) + q(x)\,y(x) = 0$$

How many independent solutions does this eq have?

Assume y_1 and y_2 are solns. (Then $ay_1 + by_2$ is also a soln.) Define the Wronskian of these two solns as:

$$W(y_1, y_2) \equiv y_1 y_2' - y_2 y_1'$$

If W vanishes, $\dfrac{y_1'}{y_1} = \dfrac{y_2'}{y_2}$ *

This implies $\dfrac{d}{dx}(y_2/y_1) = \dfrac{y_2'}{y_1} - \dfrac{y_1' y_2}{y_1^2} = \dfrac{1}{y_1^2}(y_2' y_1 - y_1' y_2)$

$$= \frac{W}{y_1^2} = 0$$

Thus $\dfrac{y_2}{y_1} = c$, a constant.

(Alternately, integrating * yields $\log y_1 = \log y_2 + c$)

So, for $W = 0$, $y_2 = c y_1 \longrightarrow$ the two solns are not linearly independent. For linearly dpt fns, we have nonzero coefficients s.t. $Ay_1 + By_2 = 0$. Thus the Wronskian will vanish if and only if the y_1 and y_2 are linearly dependent.

If $W \neq 0$, then y_1 and y_2 are linearly independent.

In fact, given one soln y_1, W can be used to construct a second lin. indpt. soln y_2.

Suppose we know one soln y_1. Pick a pt $x = x_0$ where neither $y_1(x_0)$ nor $y_1'(x_0)$ vanish, nor are they equal.
Characterize the value of the unknown fn y_2 at x_0 in terms of the values of y_1 and y_1' at x_0.

$$y_2(x_0) = \alpha\, y_1(x_0)$$

$$y_2'(x_0) = \beta\, y_1'(x_0) \qquad \alpha, \beta \text{ constants}$$

Then the value of W at x_0 is

$$W(y_1, y_2)_0 = y_1(x_0)\, y_2'(x_0) - y_2(x_0)\, y_1'(x_0) = \underbrace{(\beta - \alpha)\, y_1(x_0)\, y_1'(x_0)}_{\neq 0} \equiv W(x_0)$$

Now consider W at pts away from x_0:

$$\frac{dW(y_1, y_2)}{dx} = y_1 y_2'' - y_2 y_1'' = y_1(-p y_2' - \cancel{q y_2}) - y_2(-p y_1' - \cancel{q y_1}) = -pW$$

$$\frac{dW}{W} = -p\,dx$$

Integrating

$$\ln W\Big|_{x_0}^{x} = -\int_{x_0}^{x} p(t)\,dt$$

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} p(t)\,dt\right)$$

(never 0; y_1, y_2 lin indp

To formally construct $y_2(x)$ given $y_1(x)$, use the defn of the Wronskian:

$$W(x) = y_1 y_2' - y_2 y_1' = y_1^2 \frac{d}{dx}\left(\frac{y_2}{y_1}\right)$$

$$\frac{W(x)\,dx}{y_1^2(x)} = d\left(y_2/y_1\right)$$

$$\int_{x_1}^{x} dt\,\frac{W(t)}{y_1^2(t)} = \frac{y_2(x)}{y_1(x)}$$

$$y_2(x) = y_1(x)\int_{x_1}^{x}\frac{W(t)\,dt}{y_1^2(t)}$$

This is arbitrary since changing x_1 just shifts the value of $y_2(x)$ by a constant multiple of $y_1(x)$, which is fine by linear superposition of solns.

Do y_1 and y_2 provide the full set of solutions?

Assume a third soln $y_3(x)$. Parametrize its value at $x = x_0$ in terms of y_1 and y_2.

$$y_3(x_0) = A y_1(x_0) + B y_2(x_0) = a \quad (1)$$

$$y_3'(x_0) = A y_1'(x_0) + B y_2'(x_0) = b \quad (2)$$

Solve for coefficients A and B in terms of a, b.

$$A = \frac{a y_2'(x_0) - b y_2(x_0)}{W(y_1, y_2)|_{x_0}} \qquad B = \frac{b y_1(x_0) - a y_1'(x_0)}{W(y_1, y_2)|_{x_0}}$$

Since y_3 is assumed to be a soln of the diff. eqn.,

$$y_3''(x_0) = -p(x_0) y_3'(x_0) - q(x_0) y_3(x_0)$$

$= A y_1''(x_0) + B y_2''(x_0)$ using the results above and the fact that y_1 and y_2 satisfy the d.e.

Taking successive derivates, the general result is:

$$y_3^{(n)}(x_0) = A y_1^{(n)}(x_0) + B y_2^{(n)}(x_0)$$

But we can express the fns as Taylor expansions about x_0:

$$y_3(x) = \sum_{n \geq 0} \frac{1}{n!} (x - x_0)^n \left[A y_1^{(n)}(x_0) + B y_2^{(n)}(x_0) \right]$$

which is just the sum of the Taylor exp. of y_1 and y_2

$$= A y_1(x) + B y_2(x)$$

y_3 not linearly indpt. Only 2 solutions

Before systematically considering the series soln method, look at inhomogeneous equation.

$$Ly = r(x) = y'' + py' + qy$$

Think of $r(x)$ as being a "source term" for the field $y(x)$. eg. Poisson's eqn in em

$$\nabla^2 \phi = -4\pi\rho$$

potential

charge density

The corresponding homo eq has two solns

$$Ly_1 = Ly_2 = 0$$

We can (formally) find soln to inhomo eq with them

Write our soln y as a product $y = uv$:

$$Ly = L(uv) = (uv)'' + p(uv)' + q$$
$$= v(Lu) + uv'' + (up + 2u')v' = r$$

Let's choose $u = y_1$. Then, using $Ly_1 = 0$,

$$Ly = y_1 v'' + (y_1 p + 2y_1')v' = r$$

$$v'' + \left(p + \frac{2y_1'}{y_1}\right) v' = \frac{r}{y_1} \qquad *$$

Use properties of the Wronskian. We had

$$\left(\frac{y_2}{y_1}\right)' = \frac{W}{y_1^2} \qquad \text{and} \qquad W' = -pW$$

(page 50) (page 49)

Differentiating:

$$\left(\frac{y_2}{y_1}\right)'' = \frac{W'}{y_1^2} - \frac{2Wy_1'}{y_1^3} = \frac{-pW}{y_1^2} - 2\left(\frac{y_1'}{y_1}\right)\frac{W}{y_1^2}$$

$$= -\underbrace{\left(p + 2\frac{y_1'}{y_1}\right)}\left(\frac{y_2}{y_1}\right)'$$

same factor as in * on pg 52

$$\left\{v'' + \left(p + 2\frac{y_1'}{y_1}\right)v' = \frac{r}{y_1}\right\} \times \left(\frac{y_2}{y_1}\right)'$$

$$- \quad \left\{\left(\frac{y_2}{y_1}\right)'' + \left(p + 2\frac{y_1'}{y_1}\right)\left(\frac{y_2}{y_1}\right)' = 0\right\} \times v'$$

$$\underbrace{v''\left(\frac{y_2}{y_1}\right)' - v'\left(\frac{y_2}{y_1}\right)''}_{\left(\left(\frac{y_2}{y_1}\right)'\right)^2 \frac{d}{dx}\left(\frac{v'}{(y_2/y_1)'}\right)} = \left(\frac{r}{y_1}\right)\left(\frac{y_2}{y_1}\right)'$$

$$\frac{d}{dx}\left(\frac{v'}{(y_2/y_1)'}\right) = \left(r/y_1\right)\Big/\left(y_2/y_1\right)' = \left(r/y_1\right)\Big/\left(W/y_1^2\right) = \frac{r y_1}{W}$$

Integrating (dummy variable introduced):

$$\frac{v'}{(y_2/y_1)'} = \int dt\, \frac{r y_1}{W}$$

$$v' = \frac{d}{dx}\left[\left(\frac{y_2}{y_1}\right)\int dt\, \frac{r y_1}{W}\right] - \left(\frac{y_2}{y_1}\right)\left(\frac{r y_1}{W}\right)$$

$$v' = \frac{-r y_2}{W} + \frac{d}{dx}\left[\left(\frac{y_2}{y_1}\right)\int dt\, \frac{r y_1}{W}\right]$$

Integrating again

$$v = -\int dt\, \frac{r y_2}{W} + \left(\frac{y_2}{y_1}\right)\int dt\, \frac{r y_1}{W} = \frac{y}{u} = \frac{y}{y_1}$$

So, finally,

$$y(x) = -y_1(x)\int^x \frac{dt\, r(t) y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)} + y_2(x)\int^x \frac{dt\, r(t) y_1(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}$$

Arbitrary choice in lower limit represents adding multiples of $y_1(x)$ and $y_2(x)$, which is fine.

$\boxed{\text{So, given solns to the corresponding homogeneous equation a particular solution to the inhomogeneous equation can be found as above.}}$

Series Solution of the Homogeneous Equation

- Expansion about a regular (ordinary) pt or a regular singular pt

If $x=a$ is a regular pt [$p(a)$, $q(a)$ finite], we can expand $p(x)$ and $q(x)$ as Taylor series about a.

Assume a series solution form also

$$y(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

Example: Hermite's eq occurs in qm study of harmonic oscillator.

$$y'' - 2x\,y' + 2\omega y = 0$$

$p(x) = -2x$
$q(x) = 2\omega$

trivial expansions

Since $x=0$ is a regular point, look for an expansion of $y(x)$ about $x=0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=0}^{\infty} n\,a_n x^{n-1} + 2\omega\sum_{n=0}^{\infty} a_n x^n = 0$$

The coefficient of each power in x must vanish.

x^0: $2a_2 + 2\omega a_0 = 0$ $\boxed{a_2 = -\omega a_0}$

x^1: $6a_3 - 2a_1 + 2\omega a_1 = 0$ $\boxed{a_3 = \frac{1}{3}(1-\omega)a_1}$

x^{m-2}: $m(m-1)a_m - 2a_{m-2}(m-2) + 2\omega a_{m-2} = 0$

$$\boxed{a_m = \frac{2a_{m-2}\left[(m-2)-\omega\right]}{m(m-1)}}$$

Recursion relation gives a_m in terms of a_{m-2}.

The series in odd powers and in even powers give the two solns of Hermite's eq., $y_1 = a_0 + \cdots$
$y_2 = a_1 x + \cdots$

(The diff. operator $L = \frac{d^2}{dx^2} - 2x\frac{d}{dx} + 2\omega$ is even under $x \to -x$ so the solns have well defined parity.)

General soln $y = a\underset{\text{even}}{y_1} + b\underset{\text{odd}}{y_2}$ found using b.c.'s.

For even soln: $a_m = \frac{2\left[(m-2)-\omega\right]}{m(m-1)} \times \frac{2\left[(m-4)-\omega\right]}{(m-2)(m-3)} a_{m-4}$

$$\vdots\; = (-2)^{m/2} \frac{\omega(\omega-2)(\omega-4)\cdots(\omega-m+2)a}{m!}$$

In the case that $\omega = 2n$, where n is an integer, then for $(m-2) = 2n$ the coefficient a_m will vanish. Thus the series terminates at a_{2n}. The even series solution is then a polynomial of order $2n$ – Hermite polynomials

- Expansion about regular singular point

Near a regular singular pt $x=a$ [$(x-a)p(a)$ and $(x-a)^2 q(a)$ are finite but either $p(a)$ or $q(a)$ are not], a Taylor series expansion of $p(x)$ and $q(x)$ may not exist. So for $y(x)$ modify the approach and assume a form

$$y(x) = (x-a)^{\alpha} \sum_{n=0}^{\infty} a_n (x-a)^n$$

(Note: Wyld normalizes $a_0 = 1$)

α may be any number, positive or negative, integer or noninteger, real or complex.

This modified approach is the Frobenius method. Based on our d.e., we define new fns $P(x)$ and $Q(x)$ as follows:

$$(x-a)^2 [y'' + py' + qy] = 0$$

$$(x-a)^2 y'' + (x-a) \underbrace{P(x)}_{=(x-a)p} y' + \underbrace{Q(x)}_{=(x-a)^2 q} y = 0 \qquad *$$

$P(x)$ and $Q(x)$ are both finite at $x=a$ and can be expanded in a Taylor series.

$$P(x) = (x-a)p(x) \equiv \sum_{n=0}^{\infty} p_n (x-a)^n$$

$$Q(x) = (x-a)^2 q(x) \equiv \sum_{n=0}^{\infty} q_n (x-a)^n$$

Note that, for an ordinary point,

$$p_0 = 0 \quad \text{and} \quad q_0 = q_1 = 0 .$$

Put the expansions for y, P, and Q into *

$(x-a)^2 y''$ $\qquad 0 = (x-a)^\alpha \left[\sum_{n=0}^{\infty} (\alpha+n)(\alpha+n-1) a_n (x-a)^n \right]$

$+$

$(x-a)Py'$ $\qquad + (x-a)^\alpha \left[\sum_{n=0}^{\infty} (\alpha+n) a_n (x-a)^n \right] \left[\sum_{m=0}^{\infty} p_m (x-a)^m \right]$

$+$

Qy $\qquad + (x-a)^\alpha \left[\sum_{n=0}^{\infty} a_n (x-a)^n \right] \left[\sum_{m=0}^{\infty} q_m (x-a)^m \right]$

The coefficient of each power of $(x-a)$ must vanish separately.

Power:	Coefficient:
α	$\alpha(\alpha-1) a_0 + \alpha a_0 p_0 + a_0 q_0 = 0$
	* $\boxed{\alpha(\alpha-1) + \alpha p_0 + q_0 = 0}$
$\alpha+1$	$a_1 \left[\alpha(\alpha+1) + (\alpha+1) p_0 + q_0 \right] + \alpha p_1 + q_1 = 0$
$\alpha+2$	$a_2 \left[(\alpha+2)(\alpha+1) + (\alpha+2) p_0 + q_0 \right] + a_1 \left[(\alpha+1) p_1 + q_1 \right] + \alpha p_2 + q_2 =$
General $(\alpha+n)$th power	$a_n \left[(\alpha+n)(\alpha+n-1) + p_0(\alpha+n) + q_0 \right] + a_{n-1} \left[\; - - - \; \right] + \text{etc.} = 0$

a_0 is arbitrary so set it to 1 henceforth

The equation *, $\alpha(\alpha-1) + \alpha p_0 + q_0 = 0$, corresponding to the lowest power is a quadratic eqn in α, the indicial equation.

Generally, it has two roots, α_1 and α_2.

Putting each of these roots into the successive equations, one solves for $a_1, a_2, \ldots, a_n, \ldots$.

For each root α_i, one generates a series solution for $y(x)$:

$$y_1 = (x-a)^{\alpha_1} u_1$$

$$y_2 = (x-a)^{\alpha_2} u_2$$

u_1 and u_2 are analytic at $x=a$ (expanded as Taylor series)

For the case $x=a$ a regular point, $p_0 = q_0 = 0$, so the indicial equation reduces to $\alpha(\alpha-1) = 0$. Thus $\alpha = 0$ and $\alpha = 1$ are the two roots of the indicial equation

Let's do one straightforward example where we obtain a linearly indpt. solns near a regular singular point via this method.

Solve the hypergeometric equation

$$(x^2 - x)y'' + (2x - \tfrac{1}{2})y' + \tfrac{1}{4}y = 0$$

as a series in powers of x.

Identify $p(x) = \dfrac{(2x - \frac{1}{2})}{x(x-1)}$ and $q(x) = \dfrac{1}{4x(x-1)}$

$\therefore$ Regular singular points at $x=0$ and $x=1$.
We are seeking the expansions about $x=0$.

Assume form $y = x^p \sum_{n=0}^{\infty} a_n x^n$

Differentiating and plugging into the equation:

$$\sum_{n=0}^{\infty} (p+n)(p+n-1) a_n x^{p+n} - \sum_{n=0}^{\infty} (p+n)(p+n-1) a_n x^{p+n-1}$$

$$+ 2 \sum_{n=0}^{\infty} (p+n) a_n x^{p+n} - \frac{1}{2} \sum_{n=0}^{\infty} (p+n) a_n x^{p+n-1}$$

$$+ \frac{1}{4} \sum_{n=0}^{\infty} a_n x^{p+n} = 0$$

The coefficient of each power of x must vanish separately.

Power:

Coefficient:

x^{p-1}

($n=0$ in 2nd + 4th terms)

$$-a_0\, p\left(p-\tfrac{1}{2}\right)=0$$

This is just the indicial eqn. Don't want $a_0=0$ since this would doom successive a_i's to vanish + form a trivial soln. $\therefore$ $p=0$ and $p=\frac{1}{2}$ supply our two series solutions.

x^{m+p}

($m=n$ in 1st, 3rd, 5th and $m=n-1$ in 2nd, 4th)

$$a_{m+1}=\frac{\left[(m+p)(m+p+1)+\frac{1}{4}\right]a_m}{(m+p+1)(m+p+1/2)}$$

This is our general recursion relation for the coefficients, with the two particular results corresponding to $p=0,\frac{1}{2}$

$p=0$ case:

$$a_{m+1}=\frac{1}{2}\,\frac{(2m+1)}{(m+1)}\,a_m$$

$$=\frac{1}{2}\,\frac{(2m+1)}{(m+1)}\,\frac{1}{2}\,\frac{(2m-1)}{m}\,a_{m-1}$$

$$\vdots$$

$$=\frac{1}{2^{m+1}}\,\frac{(2m+1)!!}{(m+1)!}\,a_0$$

$$y_1=a_0\sum_{n=0}^{\infty}\frac{1}{2^n}\,\frac{(2n-1)!!}{n!}\,x^n$$

arbitrary: determined by boundary conditions

$p = \frac{1}{2}$ case: $$a_{m+1} = \frac{2(m+1)}{(2m+3)} a_m$$

$$= \frac{2(m+1)}{(2m+3)} \frac{2(m)}{(2m+1)} a_{m-1}$$

$$\vdots$$

$$= 2^{m+1} \frac{(m+1)!}{(2m+3)!!} a_0$$

$$y_2 = \sqrt{x}\, a_0 \sum_{n=0}^{\infty} 2^n \frac{n!}{(2n+1)!!} x^n$$

Very nice – two series solutions – encountered no problems.

BUT what if the indicial equation has only 1 root? Only one solution will be generated by this method. In fact, we will frequently find only one solution also in the case that the difference of the two roots is an integer.

$$\alpha_1 - \alpha_2 = s \qquad s, \text{ integer}$$

The indicial eq is $\alpha(\alpha-1) + p_0\alpha + q_0 = 0$. Put $\alpha_1 = \alpha_2 + s$ into this equation:

$$\left[(\alpha_2+s)(\alpha_2+s-1) + p_0(\alpha_2+s) + q_0\right] = 0$$

But notice that $[\,] = 0$ is the coefficient of the a_s term in the $(\alpha+s)$th power expansion (see page 58)

Because this coefficient vanishes, we cannot solve that $(\alpha+s)$th power equation for a_s.

So the method fails, typically and only α_1 leads to a series solution.

(If the other terms in the $(\alpha+s)$th eqn also vanish, we do get 2 solns $\rightarrow$ one from α_1 and the second with an arbitrary constant a_s.)

In the event that this Frobenius series method fails, how do we get the second solution?

Formally, we revert to our previous result that, given y_1 and the Wronskian W, we can find y_2 as

$$y_2(x) = y_1(x) \int_{x_1}^{x} \frac{W(t)\,dt}{y_1^2(t)} \qquad *$$

where $\ln W = -\int p\,dt$

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} p(t)\,dt\right)$$

For the expansions about a regular singular pt, we had introduced

$$P(x) = (x-a)\,p(x) = \sum_{n=0}^{\infty} p_n (x-a)^n$$

So $\quad p(t) = \dfrac{p_0}{(t-a)} + p_1 + p_2(t-a) + \cdots$

Thus $\displaystyle\int_{x_1}^{x} p(t)\,dt = p_0 \ln(x-a) + p_1(x-a)$

arbitrary

$\qquad + \dfrac{p_2}{2}(x-a)^2 + \cdots + $ constant terms from lower limit

Thus,

$$W(x) = W(x_0)\exp-\Big[\underbrace{p_0 \ln(x-a)}_{\ln(x-a)^{p_0}} + p_1(x-a) + \cdots\Big]$$

$$= W(x_0)\,e^{-\text{constant}} \exp\left(-\ln(x-a)^{p_0}\right)\exp\left(-p_1(x-a)\right)\cdots$$

$$= \frac{\text{const}}{(x-a)^{p_0}} \exp\Big[-p_1(x-a) - \frac{p_2}{2}(x-a)^2 \cdots\Big]$$

Sub this expression for W and the result for $y_1 = (x-a)^{\alpha_1} u_1$ into the expression * for y_2.

analytic fn, expressed as power series

$$y_2 \propto y_1 \int \frac{W}{y_1^2}$$

$$y_2(x) = \text{const.}\ y_1(x) \int_{x_1}^{x} dt \frac{\exp\left[-p_1(t-a) - p_2\frac{(t-a)^2}{2} + \ldots\right]}{(t-a)^{p_0+2\alpha_1}\, u_1^2}$$

$$\underbrace{\frac{\exp\left[-p_1(t-a) - p_2\frac{(t-a)^2}{2} + \ldots\right]}{u_1^2}}_{f(t)}$$

$f(t)$ is also an analytic function so express it as a Taylor series

(providing the x^0 term in u_1 does not vanish: ie. $u_1 = a_0 + a_1 x + \cdots$)

$$f(t) = \sum_{n=0}^{\infty} f_n (t-a)^n$$

The indicial equation was

$$\alpha(\alpha-1) + p_0\alpha + q_0 = \alpha^2 + \alpha(\underline{p_0 - 1}) + q_0 = 0$$

But this can be reexpressed in term of the two roots as

$$(\alpha - \alpha_1)(\alpha - \alpha_2) = 0 = \alpha^2 - \alpha(\underline{\alpha_1 + \alpha_2}) + \alpha_1\alpha_2$$

Thus we can identify $1 - p_0 = \alpha_1 + \alpha_2$.
We also have the difference $\alpha_1 - \alpha_2 = s$, integer.

So $$\frac{1}{(t-a)^{p_0+2\alpha_1}} = \frac{1}{(t-a)^{1+\alpha_1-\alpha_2}} = \frac{1}{(t-a)^{1+s}}$$

So we have

$$y_2(x) = \text{const} \cdot y_1(x) \int_{x_1}^{x} dt \, \frac{1}{(t-a)^{1+s}} \sum_{n=0}^{\infty} f_n (t-a)^n$$

where $s = \alpha_1 - \alpha_2$ is an integer.

We can now integrate this term by term:

$$y_2(x) = \text{const} \cdot y_1(x) \Big\{ f_0 \left(-\frac{1}{s}\right) \frac{1}{(x-a)^s} - \frac{f_1}{(s-1)} \frac{1}{(x-a)^{s-1}}$$

$$\cdots + f_s \ln(x-a) + f_{s+1}(x-a) + \cdots + \text{constant from lower limit} \Big\}$$

Finally,

$$\boxed{y_2(x) = \text{const} \cdot f_s \cdot y_1(x) \ln(x-a) + (x-a)^{\alpha_2} \sum_{n=0}^{\infty} g_n (x-a)^n}$$

This logarithmic term was not generated in the Frobenius series method

*

In practice, find y_1. Then assume a second solution of this form and plug it back into the equation to find the coefficients g_n.

Let's summarize and then do one example of this type.

Series Solution Summary

1. If seeking a series solution of the d.e. $y'' + py' + qy = 0$ expanded about an ordinary point $x = a$ (p and q are both analytic at $x = a$), both solutions will be analytic and take the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

The coefficients a_n satisfy a recursion relation that determines all a_n in terms of a_0 and a_1. Two linearly independent solutions result.

2. If seeking a series soln of $y'' + py' + qy = 0$ about a regular singular point $x = a$ (where $p = P/(x-a)$ and $q = Q/(x-a)^2$ with P and Q both analytic), try an expansion of the form

$$y(x) = (x-a)^{\alpha} \sum_{n=0}^{\infty} a_n (x-a)^n$$

The a_n satisfy a recursion relation and the power α satisfies an indicial equation quadratic in α: $(\alpha - \alpha_1)(\alpha - \alpha_2) = 0$

If $\alpha_1 - \alpha_2 \neq$ integer, one obtains 2 linearly indpt solns, associated with α_1 and α_2, respectively

If $\alpha_1 = \alpha_2$ and, usually, if $\alpha_1 - \alpha_2 =$ integer only one solution will arise.

The second solution can be constructed by trying an expansion of the form

$$y_2(x) = A\, y_1(x) \ln(x-a) + \sum_{n=0}^{\infty} b_n (x-a)^{n+\alpha_2}$$

3. Other possibilities (worse singularities) are more pathological.

Example Find solutions of Bessel's equation of order 0 using series expansion in x.

$$xy'' + y' + xy = 0$$

$$p(x) = \frac{1}{x} \qquad q(x) = 1$$

Thus, $x=0$ is a regular singular point.

Try $y = x^\alpha \sum_{n=0}^{\infty} a_n x^n$

The equation becomes

$$\sum_n (\alpha+n)(\alpha+n-1) a_n x^{\alpha+n-1} + \sum_n (\alpha+n) a_n x^{\alpha+n-1}$$

$$+ \sum_n a_n x^{\alpha+n+1} = 0$$

The lowest power in this expansion is $x^{\alpha-1}$, corresponding to $n=0$ in the first and 2nd terms. This yields the indicial eqn $\alpha^2 a_0 = 0 \Rightarrow \alpha = 0$ double root

To find y_1, put $\alpha = 0$ into the d.e. It reduces to:

$$\sum_n n^2 a_n x^{n-1} + \sum_n a_n x^{n+1} = 0$$

The coefficient of each power must vanish:

Power:	Coefficient:
x^0	$a_1 = 0$
x^1	$4a_2 + a_0 = 0$
x^{n-1}	$a_n = -\dfrac{a_{n-2}}{n^2} = \left(\dfrac{-1}{n^2}\right)\left(\dfrac{-1}{(n-2)^2}\right) a_{n-4}$
	$\vdots = \dfrac{(-1)^{n/2} a_0}{n^2 (n-2)^2 \cdots (2)^2}$

recursion relation

Only even powers arise in this series.

$n = \text{even} \equiv 2k \longrightarrow \quad a_{2k} = \dfrac{(-1)^k}{(2)^{2k}(k!)^2}$

$$y_1 = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2)^{2k}(k!)^2} \qquad = J_0(x)$$

Making the conventional choice $a_0 = 1$, this function is the Bessel fn of order 0

Next try to find a solution in the form including the ln factor. (generalized Frobenius form)

$$y_2 = (\ln x)\, y_1 + \underbrace{\sum_n b_n x^{n+r}}_{\equiv v}$$

Differentiating

$$y_2' = \frac{1}{x} y_1 + \ln x\, y_1' + v'$$

$$y_2'' = -\frac{y_1}{x^2} + \frac{2}{x} y_1' + \ln x\, y_1'' + v''$$

and putting into the d.e. $x y_2'' + y_2' + x y_2 = 0$:

$$\ln x \underbrace{\left[x y_1'' + y_1' + x y_1 \right]}_{=0 \text{ since } y_1 \text{ satisfies the d.e.}} + \left\{ \left[x v'' + v' + x v \right] + 2 y_1' = 0 \right\}$$

So we must find the function v such that

$$x v'' + v' + x v = -2 y_1'$$

Use $y_1 = \sum_{n=0}^{\infty} a_n x^n$ Known

$v = \sum_{n=0}^{\infty} b_n x^{n+r}$ solve for these

There is some "wishy-washy"-ness in the choice of r. According to page 68, it should be α_2. But that came from * on pg 66 where the factor of $(x-a)^{\alpha_2}$ was somewhat arbitrary.

So, to complete this example, let's choose $r = \alpha_2 = 0$. However, I have also done it using $r = 1$ and $r = 2$ and the method works.

For $r = 0$ $\quad v = x^0 \sum_{m=0}^{\infty} b_m x^m$

Expanding our eq for v:

$$0 + b_1 + \sum_{m=2}^{\infty} \left[m^2 b_m + b_{m-2} \right] x^{m-1} = -2 \sum_n n a_n x^{n-1}$$

Only odd powers of x arise in y_1 (i.e. n, even)

x^0: $(n=1)$ $\quad a_1 = 0 \Rightarrow b_1 = 0$

For $m \geq 2$, $\quad b_m = \dfrac{-2 m a_m - b_{m-2}}{m^2}$

$\therefore b_{odd} = 0$

Conventionally, we've set $a_0 = 1$. Assume our series for v starts with b_0.

$a_2 = -\frac{1}{4}$

$a_4 = \frac{1}{2^6}$ ....

$$b_2 = \frac{1}{(2)^2} - \frac{b_0}{(2)^2}$$

$$b_4 = \frac{-3}{2^7} + \frac{b_0}{(2)^6}$$

$$b_6 = \frac{11}{2^9 3^3} - \frac{b_0}{2^6 6^2} \quad \cdots$$

$$v = \left[\frac{1}{(2)^2} x^2 - \frac{3}{2^7} x^4 + \frac{11}{2^9 3^3} x^6 + \cdots \right]$$
$$+ b_0 \left[-\frac{1}{(2)^2} x^2 + \frac{1}{2^6} x^4 - \frac{1}{2^6 6^2} x^6 \cdots \right]$$

$\{\} = y_1$

So $y_2 = y_1 \ln x + v$

$y_1 = J_0$

v here was

$\left[\frac{1}{(2)^2}x^2 - \frac{3}{2^7}x^4 + \frac{11}{2^9 3^3}x^6 + \cdots\right]$

$+ \text{const} \cdot (y_1)$

Since any superposition of solns is also a soln, this last multiple of y_1 is not important. Choosing a different value of r, one will simply get different additional multiples of y_1 (eg. for $r=2$, one only gets the term $[\cdots]$)

$y_2 = J_0 \ln x + \underset{[\cdots]}{\underset{\parallel}{v}} = Y_0$

Bessel function of second kind of order 0.

Conventional to often write the second solution as

$$N_0(x) = \frac{\pi}{2} Y_0(x) + (\gamma - \log 2) J_0(x)$$

Neumann function of order 0

Euler-Mascheroni constant.