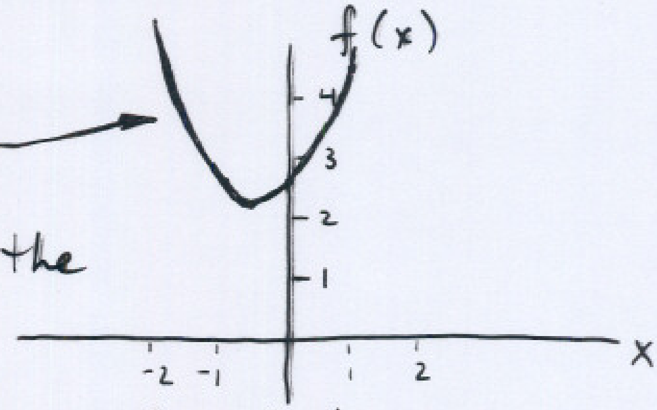
Complex analysis

The complex numbers are an extension of the real number system. They allow us, for instance, to solve a broader class of polynomial equations, consistent with the usual methods.

eg. $f(x) = x^2 + x + \frac{5}{2}$



does not intersect the x-axis so

$x^2 + x + \frac{5}{2} = 0$ has no real solutions

Define the complex number z as an ordered pair of real numbers (a, b). The "rules" for manipulating complex numbers are,

for $z_1 = (a, b)$ and $z_2 = (c, d)$:

$$z_1 \pm z_2 \equiv (a \pm c, b \pm d)$$ addition/subtractio

$$\alpha z_1 \equiv (\alpha a, \alpha b)$$

$$z_1 z_2 \equiv (ac - bd, bc + ad)$$ multiplication

Define the complex conjugate of z as z^*:

$$z^* = (a, -b)$$

Then, according to the rule for multiplicati

$$z z^* = (a^2 + b^2, 0) \qquad \text{real number}$$

Defining the modulus, $|z|$, of z as

$$|z| = +\sqrt{|z|^2} = +\sqrt{z z^*} = +\sqrt{a^2+b^2}$$

$|z|$ is nonnegati real #

$|z| = 0$ iff $z = (0,0)$

allows the rule for division to be expressed:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{z_2^*}{z_2^*} = \frac{z_1 z_2^*}{|z_2|^2} = \left(\frac{ac-bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2} \right)$$

With the rules for addition and multiplication of complex numbers, the fundamental laws of algebra follow:

1. Commutative and Associative laws of addition

$$z_1 + z_2 = z_2 + z_1$$

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

2. Commutative and Associative laws of multiplication

$$z_1 z_2 = z_2 z_1$$
$$z_1 (z_2 z_3) = (z_1 z_2) z_3$$

3. Distributive law $\quad (z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

With the above, we see that the square of $z = (0, 1)$ is $z^2 = (-1, 0) \equiv -1$.

$$z = (0, 1) = \sqrt{-1} \equiv i$$

Note for $z = i$, $z^* = -i$
so $\frac{1}{z} = \frac{1}{i} = \frac{1}{i}\frac{z^*}{z^*} = -i$

So we can switch to the usual notation

$$z = a + ib = \mathrm{Re}\, z + i\, \mathrm{Im}\, z$$

with $i^2 = -1$.

Return to the quadratic function, now considered as a fn of the complex variable z:

$z^2 + z + \frac{5}{2} = 0$ <u>does</u> have solutions given via the usual quadratic formula as $z_\pm = \frac{1}{2}\left[-1 \pm \sqrt{-9}\right]$

$$= \frac{1}{2} \pm \frac{3i}{2}$$

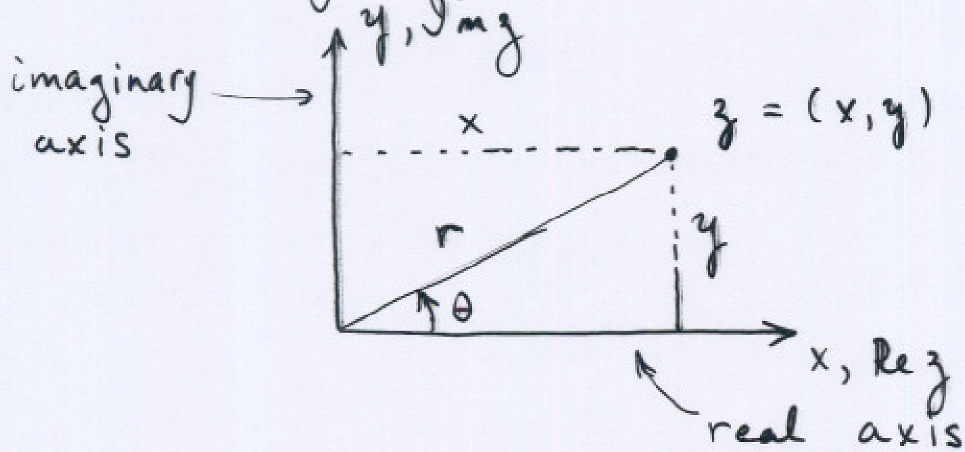
So with $z = x + iy$,

$$f(z) = z^2 + z + \frac{5}{2} = (x^2 - y^2 + 2ixy) + (x + iy) + \frac{5}{2}$$

$$= \underbrace{\left[x^2 - y^2 + x + \frac{5}{2}\right] + i\left[2xy + y\right]}_{\text{These functions}} = 0$$

simultaneously vanish at the points $(x, y) = (-\frac{1}{2}, \frac{3}{2})$ and $(-\frac{1}{2}, -\frac{3}{2})$

Because the complex numbers are represented by an ordered pair, they yield a 1-1 correspondence with points in a 2-dimensional plane. The x and y coordinates represent the real and imaginary parts of the number, respectively



Complex plane
or
Argand diagram

$$\underbrace{z = x + iy}_{\text{Cartesian coordinates}} = \underbrace{r\cos\theta + i\, r\sin\theta}_{\text{polar coordinates}} = r(\cos\theta + i\sin\theta)$$

where $r = \sqrt{x^2 + y^2} = |z|$ — · modulus, · amplitude, · absolute value } of z

$\theta = \tan^{-1}(y/x)$ — · phase, · argument } of z

Notice that $\dfrac{\partial z}{\partial \theta} = r(-\sin\theta + i\cos\theta) = iz$

$$\frac{\partial z}{\partial \theta} = iz \xrightarrow[\text{integrating}]{} z = f(r)\, e^{i\theta}$$

$f(r)$ is readily obtained by considering $\theta = 0$.

$$\boxed{z = r(\cos\theta + i\sin\theta)} = f(r)e^{i\theta}$$

$$z(\theta = 0) = r = f(r)$$

$$\boxed{z = x + iy = re^{i\theta}}$$ ← polar form of z (also get via power series)

Multiplication and division proceed very easily in polar form

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

We identify $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ Euler's formula

Changing the sign of the angle $e^{-i\theta} = \cos\theta - i\sin\theta$.

These yield the well Known relations

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Comparing these expressions with the usual defn of the hyperbolic functions

$$\cosh x = \frac{e^{x}+e^{-x}}{2} \qquad \sinh x = \frac{e^{x}-e^{-x}}{2}$$

we see that the trigonometric functions can be expressed as hyperbolic fns with an imaginary argument as

$$\cos\theta = \cosh(i\theta)$$

$$i\sin\theta = \sinh(i\theta)$$

Conversely, $\cos(i\theta) = \cosh\theta$

$$\sin(i\theta) = i\sinh\theta$$

Note that the triangle inequality holds.

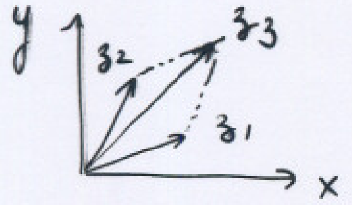
$$|z_1+z_2| \le |z_1| + |z_2|$$

This follows on looking at the square

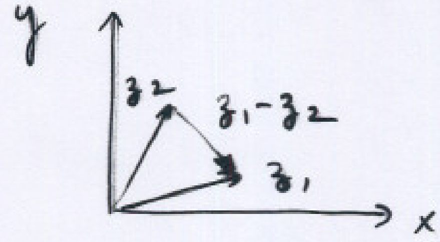
$$|z_1+z_2|^2 = (z_1+z_2)(z_1^*+z_2^*) = |z_1|^2 + |z_2|^2 + \underbrace{z_1 z_2^* + z_1^* z_2}_{\substack{=2\,\mathrm{Re}\, z_1 z_2^* \\ =2|z_1||z_2|\underbrace{\cos(\theta_1-\theta_2)}_{\le 1}}}$$

$$\le |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1|+|z_2|)^2$$

Alternately it follows geometrically on recognizing that addition of complex numbers is equivalent to addition of vectors in the complex plane.



Similarly $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$



By induction, the triangle inequality extends to any finite number of terms.

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

$\boxed{\text{Analytic Functions}}$

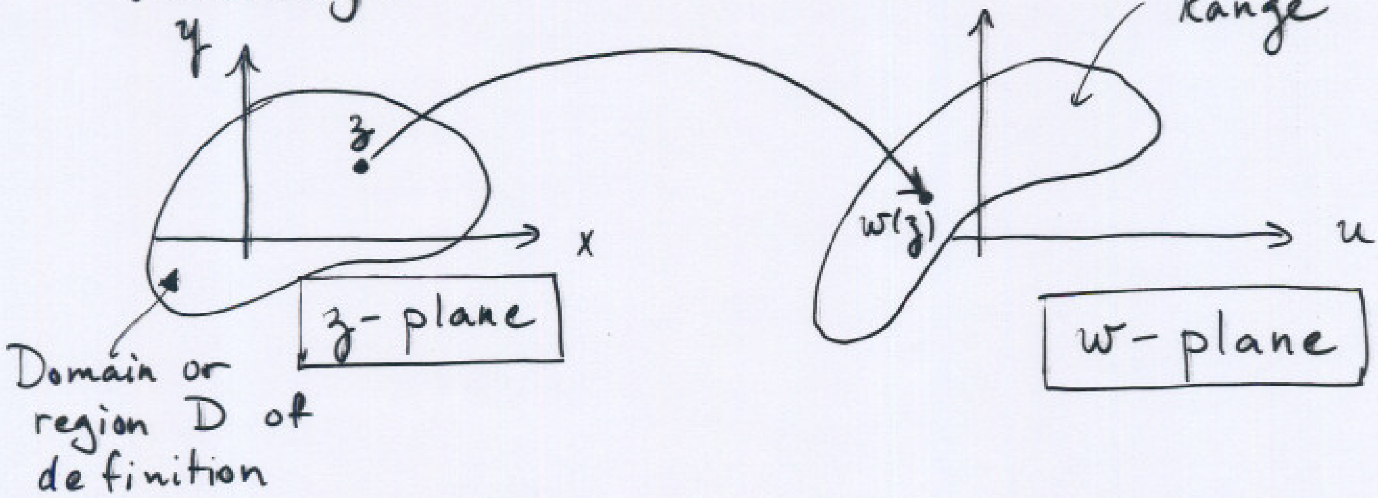
We considered the polynomial

$$f(z) = z^2 + z + \frac{5}{2} = \underbrace{\left[x^2 - y^2 + x + \frac{5}{2}\right]}_{u(x,y)} + i\underbrace{\left[2xy + y\right]}_{v(x,y)}$$

In general, a complex function of a complex argument $w(z)$ maps the # $z = x + iy$ into a new complex number $w = u + iv$ where $u(x,y)$ and $v(x,y)$ are real functions of the components of z.

We can't plot a complex fn in usual fashion using a 2 or 3-dimension plot. This is because we need a plane to specify a single complex #. We can use two planes showing the mapping of one complex plane into a copy of itself via the function.

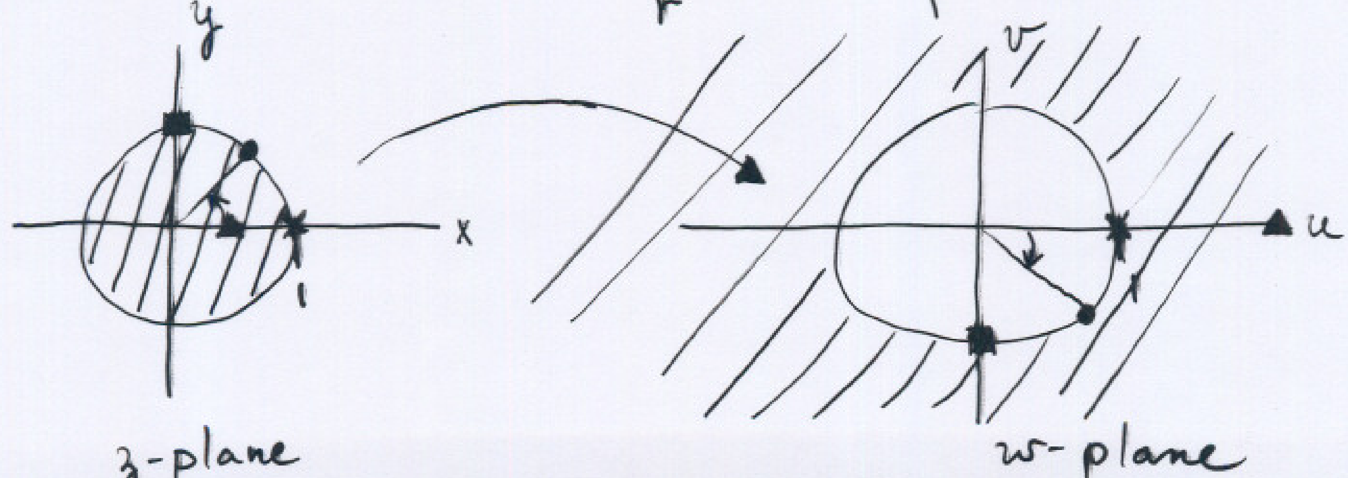
Generally:



Example : $w = \frac{1}{z} \equiv u + iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$

$z = re^{i\theta}$

$w = \frac{1}{r} e^{-i\theta} \equiv \rho e^{-i\theta}$



Restrict our attention to "analytic" function (holomorphic)

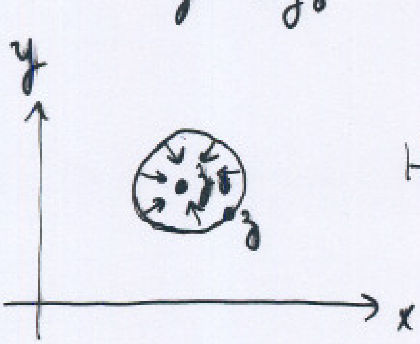
Simply stated: we must be able to differentiate w.r.t. z.

A continuous function $f(z)$ is analytic in a domain D in the complex plane if it is single-valued and differentiable everywhere in D.

We need to clarify the concept of continuity and to define differentiation w.r.t. a complex variable. But first let's note the complication that comes when we go beyond real numbers and a possible problem.

Consider a function $f(z)$. We might expect to define the derivative of $f(z)$ at a point z_0 in D as this limit:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$



How does z approach z_0?

Infinite possibilities!

Want a unique answer for the derivative.

Classic example of a function that is not analytic.

$$f(z) = |z|^2 = z z^*$$

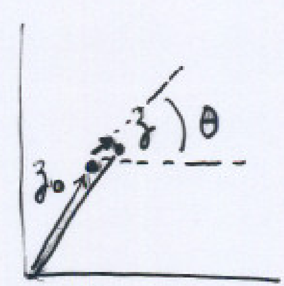
For the derivative, need $\dfrac{f(z) - f(z_0)}{z - z_0}$:

$$\frac{|z|^2 - |z_0|^2}{z - z_0} = \frac{z z^* - z_0 z_0^*}{z - z_0} = \frac{(z - z_0) z^* + z_0 z^* - z_0 z_0^*}{z - z_0}$$

$$= z^* + z_0 \frac{(z^* - z_0^*)}{(z - z_0)}$$

But $z - z_0$ is just a complex number so we can write it as $z - z_0 = |z - z_0| e^{i\theta}$. Thus $z^* - z_0^* = |z - z_0| e^{-i\theta}$

$$\therefore \frac{|z|^2 - |z_0|^2}{z - z_0} = z^* + z_0 e^{-2i\theta}$$



Explicit dependence on the direction from which z approaches z_0.

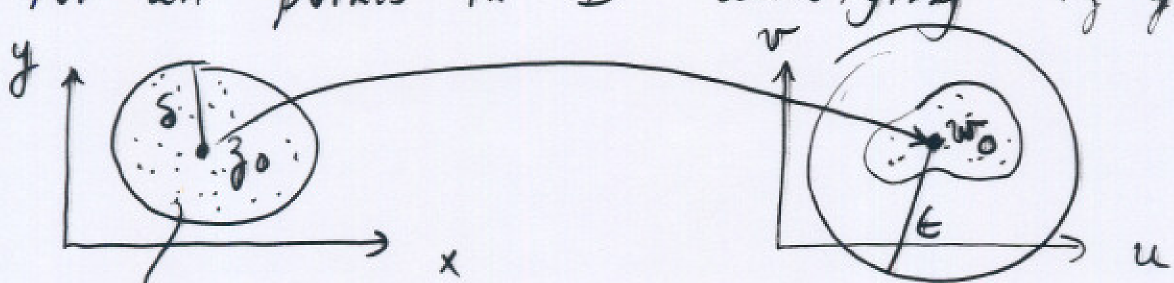
The result for the derivative is only independent of θ for $z = z_0$.

$|z|^2$ does not have a uniquely defined derivative so it is not an analytic function.

Continuity: A function $f(z)$ is continuous at z_0 if, for any given $\epsilon > 0$ (however small), we can find a number $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \qquad f(z_0) = w_0$$

for all points in D satisfying $|z - z_0| < \delta$.



neighbourhood of z_0

$z \to z_0 \qquad f(z) \to w_0 = f(z_0)$

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Differentiability: A fn. $f(z)$ is differentiable at z_0 if the limit

$$\frac{df}{dz} = f' = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$$

exists and is independent of the manner in which z tends to z_0. Alternate statement – for any $\epsilon > 0$, there exists a δ such that

$$\left| f'(z) - \frac{[f(z) - f(z_0)]}{z - z_0} \right| < \epsilon \quad \text{for } |z - z_0| < \delta$$

(Differentiable at a point – analytic in a region when differentiable at all points in the region)

Many of the theorems on differentiability in real analysis have analogs in complex analysis. The proofs are the same as in the real case. For instance –

1. A constant function is analytic.
2. $f(z) = z^n$ $(n = 1, 2, \ldots)$ is analytic.
3. The sum, product, or quotient of two analytic fns is analytic provided, in the case of the quotient, that the denominator does not vanish anywhere in the region under consideration.
4. An analytic fn. of an analytic fn. is analytic

If two functions $f_1(z)$ and $f_2(z)$ are differentiable then (a and b are constants) -

$$\frac{d}{dz}\left[a f_1(z) + b f_2(z)\right] = a f_1'(z) + b f_2'(z)$$

$$\frac{d}{dz}\left[f_1(z) f_2(z)\right] = f_1(z) f_2'(z) + f_1'(z) f_2(z)$$

$$\frac{d}{dz}\left[\frac{f_1(z)}{f_2(z)}\right] = \frac{f_2(z) f_1'(z) - f_1(z) f_2'(z)}{f_2^2(z)} \qquad \text{for } f_2(z) \neq 0$$

eg $\dfrac{d}{dz} z^n = n z^{n-1}$ $\qquad n = 1, 2, 3, \ldots$

We need to find necessary and sufficient conditio[n]
for analyticity.

Consider $f(z) = u(x,y) + i\,v(x,y)$

The derivative of $f(z)$ w.r.t. z is defined as

$$\frac{df}{dz} = f' = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \left[\frac{\Delta u}{\Delta z} + i \frac{\Delta v}{\Delta z} \right]$$

$\Delta z = \Delta x + i\Delta y$

$$f' = \lim_{\Delta z \to 0} \left[\frac{\{u(x+\Delta x, y+\Delta y) - u(x,y)\}}{\Delta z} + i \frac{\{v(x+\Delta x, y+\Delta y) - v(x,y)\}}{\Delta z} \right.$$

① Set $\Delta y = 0$ and let $z + \Delta z$ approach z along the real axis direction.

$$\frac{df}{dz} = \lim_{\Delta x \to 0} \left[\frac{\{u(x+\Delta x, y) - u(x,y)\}}{\Delta x} + i \frac{\{v(x+\Delta x, y) - v(x,y)\}}{\Delta x} \right]$$

$$= \frac{\partial u(x,y)}{\partial x} + i \frac{\partial v(x,y)}{\partial x}$$

② Now set $\Delta x = 0$ and let $z + \Delta z$ approach z along the imaginary axis direction.

$$\frac{df}{dz} = \lim_{\Delta y \to 0} \left[\frac{\{u(x, y+\Delta y) - u(x,y)\}}{i\Delta y} + i \frac{\{v(x,y+\Delta y) - v(x,y)\}}{i\Delta y} \right]$$

$$= \frac{\partial v(x,y)}{\partial y} - i\,\frac{u(x,y)}{\partial y}$$

Insisting that the two forms ① and ② of the derivative are equal and equating real and imaginary parts, we find:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Cauchy-Riemann equations

- necessary conditions for analyticity

If the four partial derivatives exist, are <u>continuous</u>, and satisfy the Cauchy-Riemann equations, then the fn $f(z)$ is analytic.

Sufficiency: By hypothesis, the partial derivatives of u and v exist and are continuous. With these assumptions, u and v must be continuous because their partial derivatives exist. From calculus of several variables, under these assumptions, it follows that —

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

and $$dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy$$

Then $$df(z) = du + i\,dv$$

$$= \frac{\partial u}{\partial x}\,dx + \underbrace{\frac{\partial u}{\partial y}}_{= -\frac{\partial v}{\partial x}}\,dy + i\left[\frac{\partial v}{\partial x}\,dx + \underbrace{\frac{\partial v}{\partial y}}_{= \frac{\partial u}{\partial x}}\,dy\right]$$

C.R. equations $\longrightarrow$

$$= \frac{\partial u}{\partial x}(dx + i\,dy) + i\frac{\partial v}{\partial x}(dx + i\,dy) = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]dz$$

$$\boxed{\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}$$

This result was derived with no reference to the manner in which $z + \Delta z$ tends to z.

Sufficient

The Cauchy-Riemann equations are equivalent to the statement that $f(z)$ depends on z only, not on z^*.

$$z = x + iy \qquad z^* = x - iy$$

$$x = \frac{1}{2}(z + z^*) \qquad y = \frac{-i}{2}(z - z^*)$$

Consider z, z^* as independent variables replacing x, Using chain rule –

$$\partial_z \equiv \frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\partial_x - \frac{i}{2}\partial_y$$

$$\partial_{z^*} \equiv \frac{\partial}{\partial z^*} = \frac{\partial x}{\partial z^*}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial}{\partial y} = \frac{1}{2}\partial_x + \frac{i}{2}\partial_y$$

$$\frac{\partial f}{\partial z^*} = \frac{\partial u}{\partial z^*} + i\frac{\partial v}{\partial z^*} = \frac{1}{2}\left\{\left[\cancel{\frac{\partial u}{\partial x}} + \frac{i}{2}\overset{= -\frac{\partial v}{\partial x}}{\frac{\partial u}{\partial y}}\right] + i\left[\cancel{\frac{\partial v}{\partial x}} + i\underset{= \frac{\partial u}{\partial x}}{\cancel{\frac{\partial v}{\partial y}}}\right]\right\} = 0$$

So, for a function satisfying the C-R equations, $\frac{\partial f}{\partial z^*} = 0$. (hence the trouble with $|z|^2$)

Differentiating the Cauchy-Riemann equations yields:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2} \qquad \therefore \nabla^2 u = 0$$

Similarly $\nabla^2 v = 0$. The real and imaginary parts of an analytic function separately satisfy Laplace's equation.

(u and v called conjugate harmonic functions)

Also note $\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right) = \left(\frac{\partial v}{\partial y}\right)\left(-\frac{\partial u}{\partial y}\right)$

This can be expressed as:

$$[\vec{\nabla} u] \cdot [\vec{\nabla} v] = 0$$

$\vec{\nabla} v$ is orthogonal to lines $v = \text{constant}$

$\vec{\nabla} u$ is orthogonal to lines of $u = \text{constant}$.

$\therefore$ $u = \text{const}$ and $v = \text{const}$ curves intersect at right angles.

A function that is analytic everywhere in the complex plane is called an <u>entire</u> function.

Some basic analytic functions -

A very useful function in the complex domain is the exponential function which we define, for $z = x + iy$ as

$$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$$

This function satisfies the C-R equations and is entire.

$$\frac{de^z}{dz} = e^z \qquad \text{(as with real numbers)}$$

Using e^z, our previous results for trig fns extend to arguments z as expected.

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\frac{d\cos z}{dz} = -\sin z \qquad \frac{d\sin z}{dz} = \cos z$$

$\cos z$ and $\sin z$ are both entire functions.

$\tan z \equiv \dfrac{\sin z}{\cos z}$ is analytic everywhere except where $\cos z = 0$.

This condition turns out to hold at the same points as $\tan x$ is ∞.

$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}) = \tfrac{1}{2}\left[e^{i(x+iy)} + e^{-i(x+iy)} \right]$$

$$= \tfrac{1}{2}\left[e^{y}(\cos x - i\sin x) + e^{-y}(\cos x + i\sin x) \right]$$

$$= \cos x \,\frac{(e^{y} + e^{-y})}{2} - i\sin x \,\frac{(e^{y} - e^{-y})}{2}$$

$$= \cos x \cosh y - i \sin x \sinh y$$

$\cos z = 0$ requires both $\cos x \overbrace{\cosh y}^{\geq 0 \text{ for all real } y} = 0 \longrightarrow x = (2n+1)\frac{\pi}{2}$

$\longrightarrow$ and $\underbrace{\sin x}_{=1} \underbrace{\sinh y}_{=0 \text{ for } y=0} = 0$

real axis

Multivalued functions: Branch points and cuts
Riemann sheets

All the functions considered so far are single-valued for a particular value of z, $f(z)$ has a unique definite value.

A multivalued function has two or more distinct values for each value of z.

We will develop a way of identifying a branch of such functions in some region such that the function is continuous and single valued. Then we can apply the theorems of analytic function theory.

Multivalued functions include:

not rational

$$\sqrt{z} \;;\; \sqrt[n]{z} \;;\; \sqrt{(z-a)(z-b)} \;;\; \log z \;;\; z^{\alpha}$$

$$\sin^{-1}(z) \;;\; \cos^{-1}(z)$$

Introduce the relevant new concepts by focussing on the classic example of $\sqrt{z}$.

$\boxed{f(z) = \sqrt{z}}$ In real analysis, there is a sign ambiguity in $\sqrt{x}$. Two values of the square root are possible for each values of x. We can make this multivaluedness explicit by using the polar form of z and recognizing its periodicity.

$z = re^{i\theta}$ is periodic in θ, with period 2π.

$$z = re^{i(\theta+2\pi)} = re^{i\theta}\underbrace{e^{i2\pi}}_{=1} = re^{i\theta}$$

$$\boxed{z(\theta) = z(\theta + 2\pi)}$$

$$f(z) = z^{1/2} = (re^{i\theta})^{1/2} = \sqrt{r}\, e^{i\theta/2} \equiv w_1$$

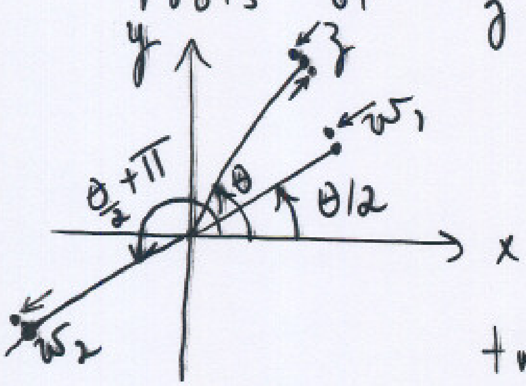
$$\downarrow \quad z \longrightarrow z(\theta+2\pi)$$

$$f(z) = (re^{i(\theta+2\pi)})^{1/2} = (\sqrt{r}\, e^{i\theta/2})\underbrace{e^{i\pi}}_{=\cos\pi + i\sin\pi = -1}$$

$$= -\sqrt{r}\, e^{i\theta/2} \equiv w_2$$

where $\sqrt{r} \equiv +\sqrt{x^2+y^2}$

So the periodicity of z provides the two roots of z.



We refer to the values

$$w_1 = \sqrt{r}\, e^{i\theta/2} \quad \text{and}$$

$$w_2 = \sqrt{r}\, e^{i(\theta+2\pi)/2} \quad \text{as the}$$

two branches of $\sqrt{z}$.

(Note: $f(z) = (re^{i(\theta+4\pi)})^{1/2} = \sqrt{r}\, e^{i\theta/2} = w_1$)

$z_1 = \cdot$ and $z_2 = \cdot$ are close together but $w_1 = \cdot$ and $w_2 = \cdot$ are not.

The C-R equations are satisfied for each branch, except at $z=0$ where the partial derivatives blow up.

eg
$$w_1 = \sqrt{r}\, e^{i\theta/2} = \sqrt{r}\left[\cos\theta/2 + i\sin\theta/2\right] = u + iv$$

$$r = \sqrt{x^2+y^2} \qquad \theta = \tan^{-1}(y/x)$$

It is a bit of work but can show that

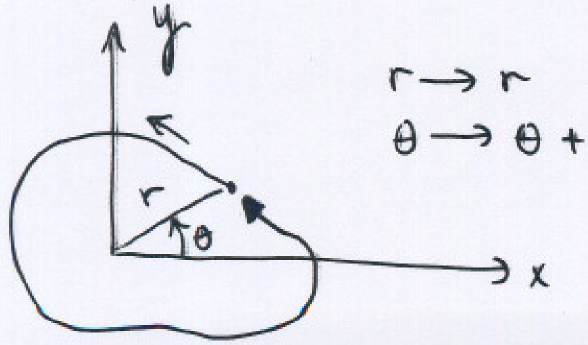
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{1}{2\sqrt{r}}\cos(\theta/2)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{1}{2\sqrt{r}}\sin(\theta/2)$$

← not defined at $r=0$.

So, if we stick to one branch, $\sqrt{z}$ is continuous, differentiable, single-valued. But if θ increases by 2π, $f(z)$ changes discontinuously from the value w_1 to w_2.

$f(z) = \sqrt{z}$ is discontinuous on completing a circuit in θ that encircles $z=0$.



$$\left.\begin{array}{l} r \to r \\ \theta \to \theta + 2\pi \end{array}\right\} w_1 \to w_2$$

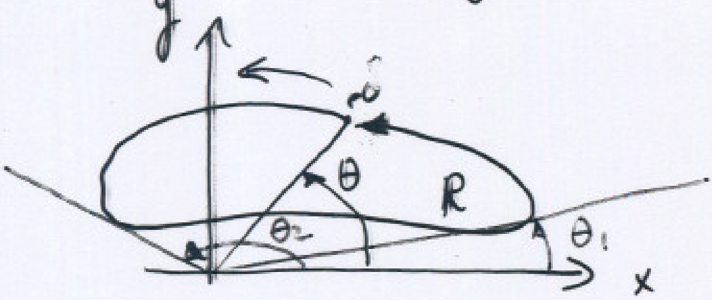
Second circuit $\left.\begin{array}{l} r \to r \\ \theta + 2\pi \to \theta + 4\pi \end{array}\right\} w_2 \to$

Return to original value of $f(z)$ with two circuits ⇒ continuous variation through both

Two circuits: $\theta \longrightarrow \theta + 4\pi$

$$f\left(re^{i(\theta+4\pi)}\right) = \sqrt{r}\, e^{\frac{i\theta}{2}} \underbrace{e^{i2\pi}}_{=1} = f\left(re^{i\theta}\right)$$

If we execute a closed circuit about a path that does not enclose the pt $z=0$, we have only single valued behaviour, no discontinuity.



Traversing the path, θ increases to θ_2, decreases to θ_1 and increases back to its original.

θ never increases by 2π. Along this path, we stay on whichever branch we start on.

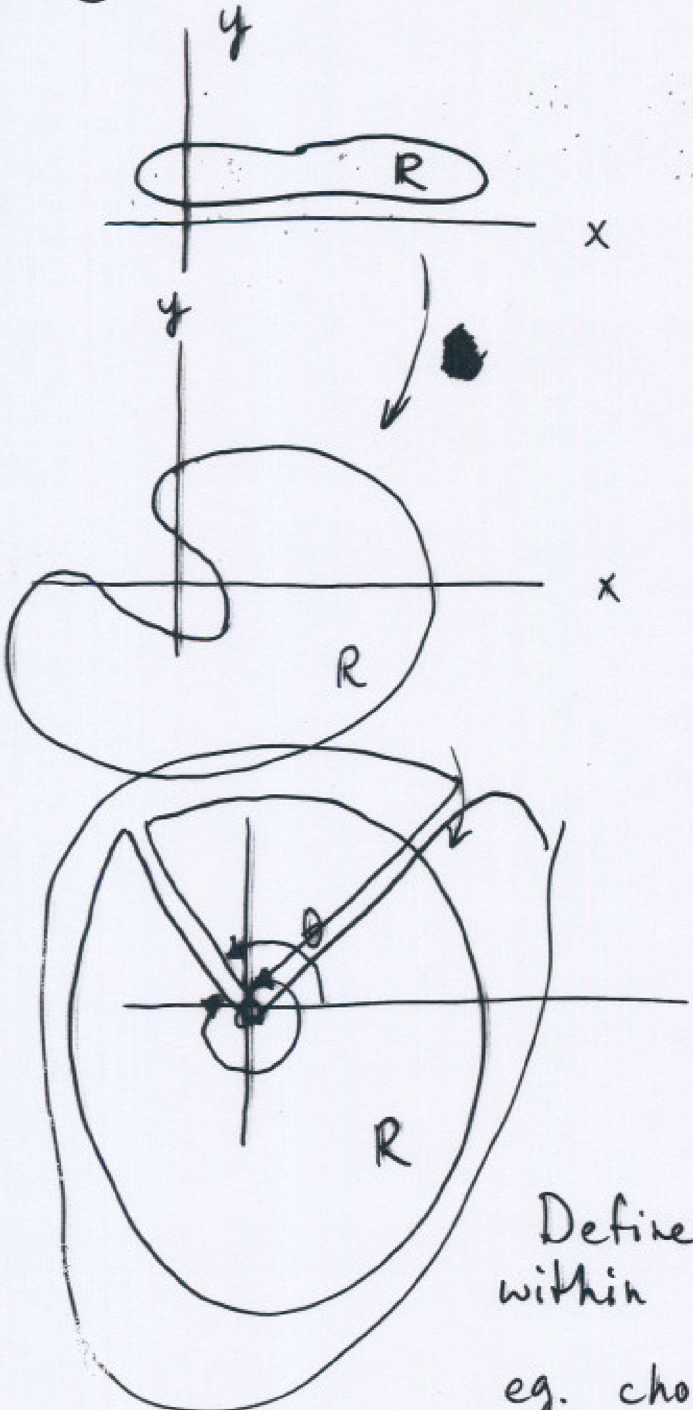
$z=0$ is called the branch point of $\sqrt{z}$. It is a type of singularity. $\sqrt{z}$ is double-valued in any region that includes $z=0$.

Dealing with double-(multi-) valued functions:

① • restrict to regions that exclude branch point (s).
$\longrightarrow$ rejects one of the branches.

② • Riemann sheets
$\longrightarrow$ single valued fn while keeping both branches by extending the space

1



R does not include the branch point $z=0$

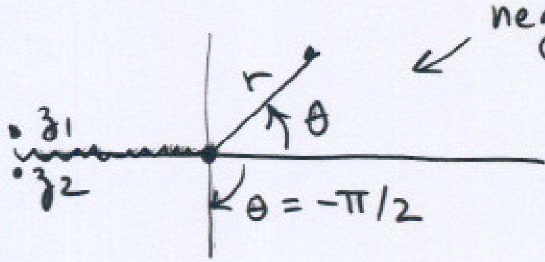
Expand R to include as much of the complex plane as possible.

A cut running from the branch point out to infinity in some direction will be left.

Choose the cut for convenience. It does not matter where you put it.

Define the function only within the range of the angle.

eg. choosing the cut along the negative real axis



$$\boxed{-\pi \le \theta \le \pi}$$

Don't cross the cut!

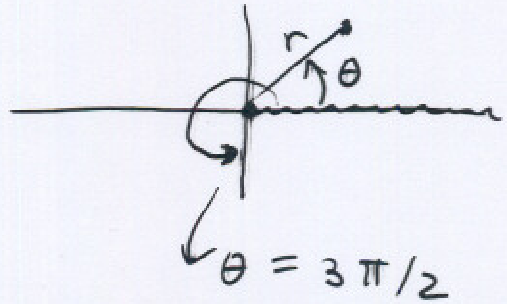
$$\sqrt{z} = \sqrt{r}\, e^{i\theta/2}$$

$$\left.\begin{aligned} z_1 &= re^{i\pi} = -r \\ z_2 &= re^{-i\pi} = -r \end{aligned}\right\} \text{equal but:} \quad \begin{aligned} \sqrt{z_1} &= \sqrt{r}\, e^{i\pi/2} = i\sqrt{r} \\ \sqrt{z_2} &= \sqrt{r}\, e^{-i\pi/2} = -i\sqrt{r} \end{aligned}$$

There is a discontinuity in $f(z) = \sqrt{z}$ as the cut is crossed. $\sqrt{z}$ is not analytic on the cut.

Alternate cut:
Restrict $\boxed{0 \le \theta < 2\pi}$

$$\sqrt{z} = \sqrt{r}\, e^{i\theta/2}$$

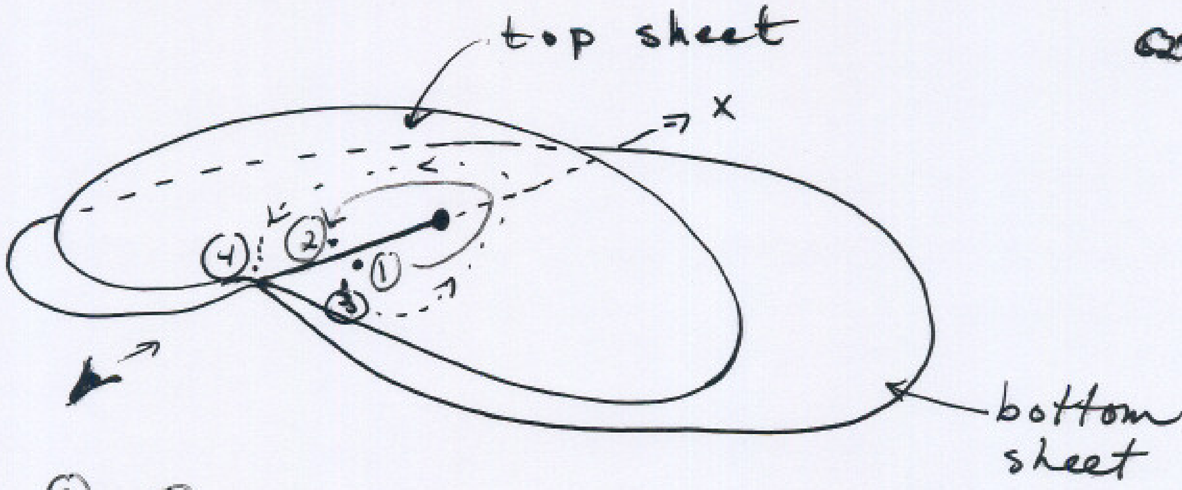


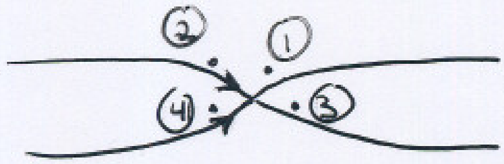
For these two cut choices, $\sqrt{z}$ agrees in the upper half plane but differs between them by a minus sign in the lower half plane.

② Riemann sheets

An elegant way of dealing with multivalued functions while allowing all values of θ was devised by Riemann. The idea is to consider the function $f(z)$ to be defined on an enlarged 2-d surface.

Make a semi-infinite cut in the complex plane extending from the branch point to ∞. Choose the negative real axis for demonstration purpose. Stack this plane with a second sheet, a copy of the complex plane, also cut and connected to the first sheet along the branch cut.





Start at 1 : $\theta = -\pi$ on top sheet.

Move in a circuit about the origin on the top sheet to 2 where $\theta = +\pi$.

At the cut, slip continuously down from the top sheet (2nd quadrant) to the bottom sheet (3rd quadrant) as θ continues to increase from $\theta = \pi$ through $\theta = 3\pi$ during the circuit about the origin on the lower sheet.

z and $f(z)$ change continuously as the circuits are made.

Start point 1 : $\theta = -\pi$

$$z = re^{-i\pi} = -r$$
$$\sqrt{z} = \sqrt{r}\, e^{-i\pi/2} = -i\sqrt{r}$$

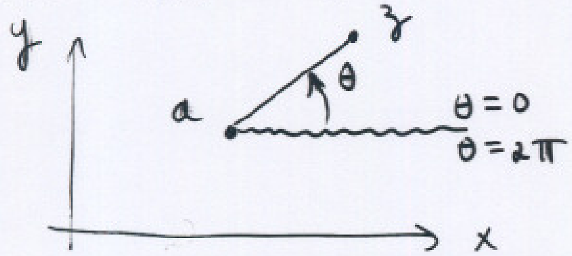
After circuit on lower sheet to 4 : $\theta = 3\pi$ move up to top sheet $z = re^{i3\pi} = -r$ $\quad \sqrt{z} = \sqrt{r}\, e^{i3\pi/2} = -i\sqrt{r}$

Back to start.

Traversing any closed path that does not encircle $z=0$, the branch point, one will never move from one sheet to the other.

No discontinuity arises for any closed path that does not include the origin. The function is single-valued on each Riemann sheet.

For $f(z)=\sqrt{z-a}$, we have the branch point at $z=a$. $f(z)$ is single valued on a 2-sheeted Riemann surface that is cut from a to infinity. For instance, choosing the cut as:



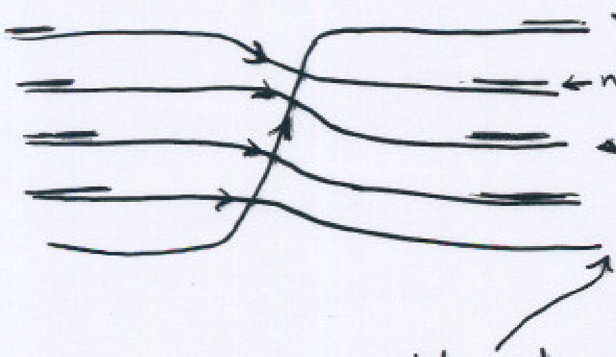
$$f(z)=\sqrt{|z-a|}\, e^{i\theta/2}$$

The nth root of z is very similar to $\sqrt{z}$.

$$f(z) = \sqrt[n]{z} = \underbrace{\sqrt[n]{r}\; e^{i(\theta+2\pi k)/n}}_{\text{the } n \text{ nth-roots.}} \qquad k=0,1,2,\dots,n-1$$

$\sqrt[n]{z}$: n-valued function

$f(z)=\sqrt[n]{z}$ is single valued on an n-sheeted Riemann surface cut from $z=0$ to infinity, eg. along negative real axis



$\leftarrow \sqrt[n]{z} = \sqrt[n]{r}\, e^{i\theta/n} \quad -\pi \le \theta < \pi$

$\leftarrow \sqrt[n]{z} = \sqrt[n]{r}\, e^{i\theta/n} \quad \pi < \theta < 3\pi$

$\leftarrow \sqrt[n]{z} = \sqrt[n]{r}\, e^{i\theta/n} \quad 3\pi < \theta < 5\pi$

$$\sqrt[n]{z} = \sqrt[n]{r}\, e^{i\theta/n} \qquad (2n-3)\pi \le \theta < (2n-1)\pi$$

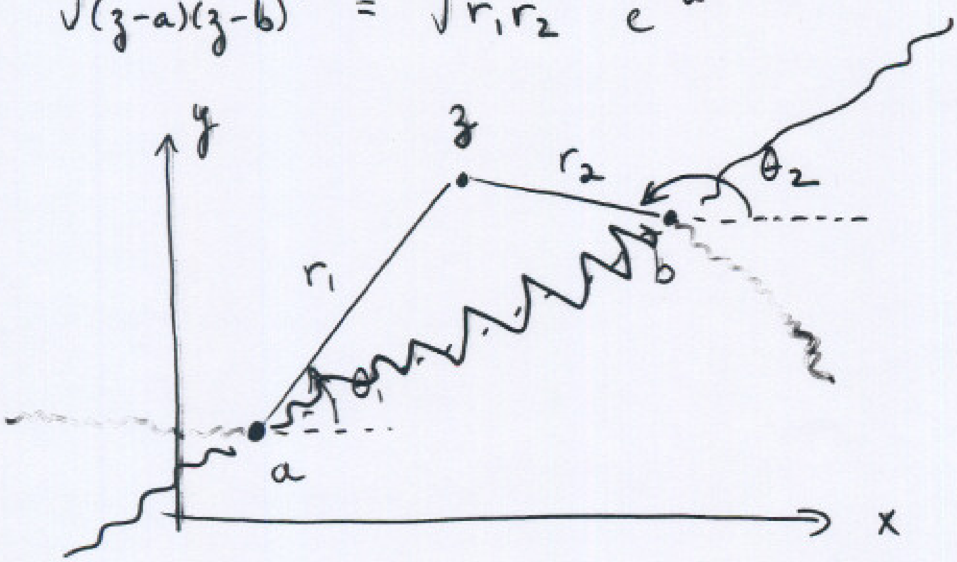
A function based on roots that has two branch points is

$$f(z) = \sqrt{(z-a)(z-b)}$$

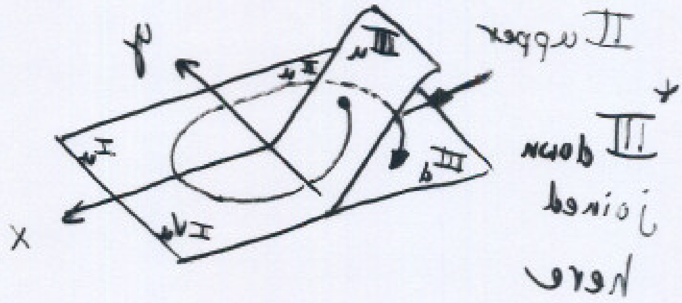
Without restricting z or devising a Riemann surface, this function is double valued. We can define a single-valued branch as

$$\sqrt{(z-a)(z-b)} = \sqrt{r_1 r_2}\; e^{\frac{i}{2}(\theta_1+\theta_2)}$$

θ_1, θ_2 such that cuts not crossed

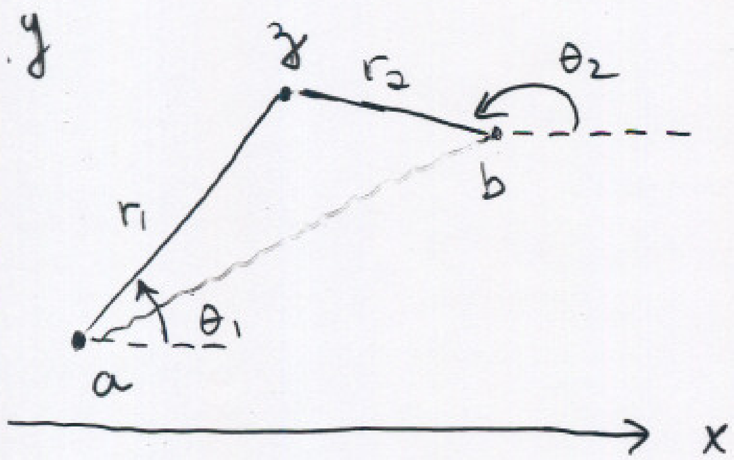


Cuts chosen arbitrarily from each branch pt to ∞.



x
y
II upper
III down
joined here

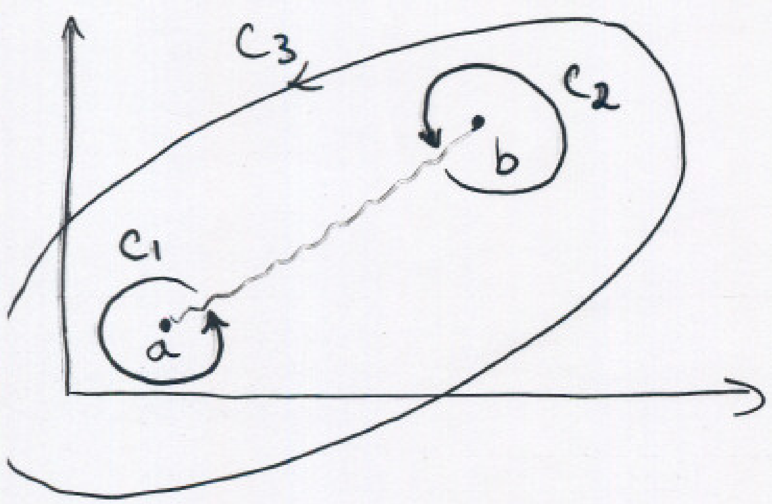
Can picture things more simply if choose the cuts along the line joining a and b.



Cut is effectively between a and b.

$$\sqrt{(z-a)(z-b)} \sim e^{i\theta_1/2} e^{\frac{i\theta_2}{2}}$$

Consider different sorts of paths:



C_1:- θ_1 increases by 2π - θ_2 moves through a range $< 2\pi$ and returns to its original value.

$$\therefore e^{\frac{i}{2}(\theta_1+\theta_2)} \longrightarrow e^{\frac{i}{2}(\theta_1+\theta_2)} \underbrace{e^{i\pi}}_{-1}$$

$\therefore$ $f(z)$ changes sign corresponding to the discontinuity along the cut

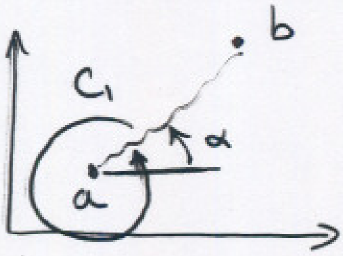
C_2: - θ_2 increases by 2π ; θ_1 returns to original value; $\therefore$ $f(z)$ changes sign.

C_3: Both θ_1 and θ_2 increase by 2π

$$e^{\frac{i}{2}(\theta_1+\theta_2)} \longrightarrow e^{\frac{i}{2}(\theta_1+\theta_2)} \cdot \underbrace{e^{i2\pi}}_{=+1} \quad \therefore f(z) \text{ unchanged}$$

From the Riemann surface perspective, we have two sheets cut and joined along the line joining a & b.

For a curve like C_3, one stays on either the top or the bottom sheet, wherever one started, so the function is continuous and single valued.



C_1: Say we start on top sheet. As θ_1 moves through α, slip down the cut to the lower sheet. θ_1 continuously increases on that sheet from α to $\alpha + 2\pi$, and comes back up through the cut to the top sheet. So the fn is continuous and single valued.

C_2: Same thing.

Another example of a multivalued function is the logarithm. We note that for a complex number z

$$z = re^{i\theta} = re^{i(\theta + 2n\pi)} \qquad 0 \le \theta \le 2\pi \qquad n = 0, \pm 1, \pm 2, \ldots$$

Thus

$$\log z = \ln r + i\theta + i2n\pi \qquad 0 \le \theta \le 2\pi$$

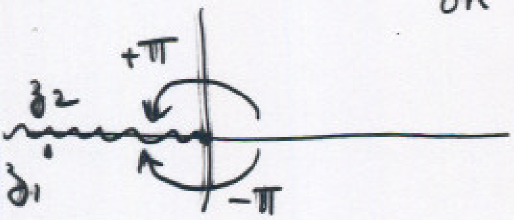
usual natural log of a real #.

Each value of n corresponds to a distinct branch of the logarithm. So this way of looking at things with θ restricted to the range $0 \rightarrow 2\pi$, we think of an infinite number of single-valued branches of the logarithm.

Alternately think of θ increasing continuously in multiple circuits about $z=0$ moving from one branch of the logarithm to the next with each circuit, yielding an infinite sets of values of the logarithm.

$\boxed{z=0 \text{ is a branch point of infinite order}}$

In the Riemann sheet picture, we consider a stack of an infinite number of sheets each cut from $z=0$ to $z=\infty$ along some path. Arbitrarily choosing the negative real axis, on one Riemann sheet $-\pi<\theta<\pi$.



At z_1 on this sheet,

$$\log z_1 = \ln r - i\pi$$

and at z_2 on this sheet $\log z_2 = \ln r + i\pi$.

There is a discontinuity across the cut of

$$\log z_2 - \log z_1 = 2\pi i$$

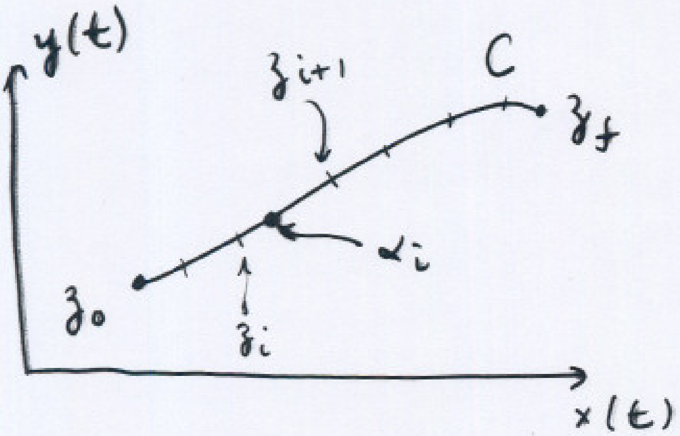
If, in continuing from z_2 across the cut, we slip to the next Riemann sheet, then θ increases continuously from $\pi \rightarrow 3\pi$, with the logarithm increasing continuously in a corresponding manner and the logarithm is singlevalued on each sheet.

For $z = re^{i\theta}$ with $0 \leq \theta < 2\pi$, the branch $\log z = \mathrm{Log}\, z = \ln r + i\theta$ is called the principal value or principal branch of the logarithm.

Integration of Functions of a Complex Variable:

Contour Integrals

Since a complex number is represented as a point in the complex plane, we might expect to define integration of a function as represented by a line integral of that function along a curve in the complex plane.



z- plane

Consider the real and imaginary parts of a complex number $z = x + iy$ to be represented parametrically:

Let t be a parameter ranging from $t_0 \to t_f$ and let $z(t)$ be a curve or <u>contour</u> C in the complex plane with endpoints $z_0 = z(t_0)$ and $z_f = z(t_n)$.

Subdivide the contour C into segments by distributing points $z_0, z_1, \ldots, z_n$ along the curve. Approximate the curve by drawing straight line segments from each $z(t_i)$ to $z(t_{i+1})$.

Define the integral of a function f of a complex variable by forming the sum

$$I_n = \lim_{|\Delta z_i| \to 0} \sum_{i=0}^{n} f(\alpha_i)\, \Delta z_i$$

function is evaluated at an arbitrary point between z_i and z_{i+1}.

The limit is taken such that ~~~~ the partition is arbitrarily fine, $|\Delta z_i| \to 0 \ \forall i$, which is equivalent to $n \to \infty$.

If the sum I_n defined above approaches a limit I as $|\Delta z_i| \to 0 \ \forall i$ —

that is, if, given an ϵ, there exists a δ such that

$$|I_n - I| < \epsilon \quad \text{for } |\Delta z_i| < \delta \ \forall i \quad —$$

then the limit I is defined as the integral —

$$I \equiv \int_C dz\, f(z)$$

We have $f(z) = u(x,y) + i v(x,y)$ and $dz = dx + i dy$:

$$\int_C dz\, f(z) = \int_C (dx + i dy)(u + i v)$$

$$= \int_C (u\,dx - v\,dy) + i \int_C (v\,dx + u\,dy) \qquad *$$

Thinking in terms of the parametric form for the contour, we have $x(t)$ and $y(t)$ so that

$$dx = \frac{dx}{dt}\,dt \qquad\qquad dy = \frac{dy}{dt}\,dt$$

$$\int_C dz\, f(z) = \int_{t_0}^{t_n} \left(u \frac{dx}{dt} - v \frac{dy}{dt} \right) dt + i \int_{t_0}^{t_n} \left(v \frac{dx}{dt} + u \frac{dy}{dt} \right) dt$$

This form very clearly shows that each term is just an ordinary one-dimensional real integral. Alternately can drop the idea of the parametrization of the curve and just recognize that, in *, integrating over dx, we condsider $y = y(x)$ and integrating over dy, we consider $x = x(y)$.

So, from either perspective, the integration in the complex plane is reduced to a combination of real one dimensional integrals.

Fundamental thm of integral calculus:

With $\dfrac{dz(t)}{dt} = \dfrac{dx(t)}{dt} + i \dfrac{dy(t)}{dt}$, the integral becomes

$$I = \int_C dz\, f(z) = \int_{t_0}^{t_n} (u + iv) \frac{dz}{dt}\, dt$$

$$= \int_{t_0}^{t_n} f[z(t)] \frac{dz(t)}{dt}\, dt$$

Assume $f(z)$ can be expressed as the derivative of a function $F(z)$ in the region of the complex plane including the contour:

$$f(z) = \frac{dF(z)}{dz}$$

$F(z)$ is the primitive of $f(z)$.

$\Rightarrow F(z)$ is analytic

For z on C, $\quad f(z)\dfrac{dz}{dt} = \dfrac{dF(z)}{dz}\dfrac{dz}{dt} = \dfrac{dF[z(t)]}{dt}$

so $\quad I = \displaystyle\int_C f(z)\,dz = \int_{t_0}^{t_n} \frac{dF[z(t)]}{dt} = F(z_f) - F(z_0)$

endpoints of contour

Because we understand how to reduce integration in the complex plane to real 1-d integrals, we also have:

$$\boxed{\int_C = -\int_{-C}}$$

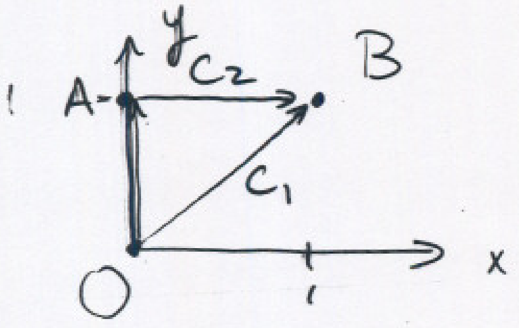
and

$$\boxed{\int_{C_1} + \int_{C_2} = \int_{C_1 + C_2}}$$

For C a closed curve that does not intersect itself ( vs ) the convention is to take $\oint_C$ as counterclockwise along the contour.

Examples

1. Integrate $f(z) = \sin z$ in the complex plane from $\underbrace{z=0}_{O}$ to $\underbrace{z = 1+i}_{B}$



Use two contours:

① $C_1 = OB$

② $C_2 = OA + AB$

$$\boxed{f(z) = u(x,y) + i\, v(x,y)}$$

$$\sin z = [\cosh y \sin x + i \sinh y \cos x]$$

$$\boxed{\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)}$$

$$\int_C \sin z\,dz = \int_C [\cosh y \sin x\,dx - \sinh y \cos x\,dy]$$

$$+ i\int_C [\sinh y \cos x\,dx + \cosh y \sin x\,dy]$$

① Along $C_1 = OB$, we have $x = y$:

$$I_1 = \int_{C_1} \sin z \, dz = \int_0^1 (1+i) \cosh x \sin x \, dx$$

$$- (1-i) \int_0^1 \sinh x \cos x \, dx$$

$$= (1 - \cosh 1 \cos 1) + i (\sinh 1 \sin 1)$$

write hyperbolic in terms of exponentials; int. by parts

Note that $\cos z = \cosh y \cos x - i \sinh y \sin x$.

Assuming $\displaystyle\int_{O=0}^{B=1+i} \sin z \, dz = -\cos z \Big|_0^{1+i} = 1 - \cosh 1 \cos 1 + i \sinh 1 \sin 1$

So we might expect we don't always have to go through this process.

② $C_2 = \underbrace{OA}_{x=0\,;\, dx=0} + \underbrace{AB}_{y=1\,;\, dy=0}$

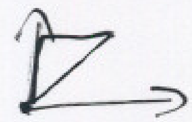
$$I_2 = \int_{C_2} \sin z \, dz = \left\{ \int_{OA} \sin z \, dz \right\} + \left[\int_{AB} \sin z \, dz \right]$$

$$= \left\{ - \int_0^1 \sinh y \, dy \right\} + \left[\int_0^1 \cosh 1 \sin x \, dx + i \int_0^1 \sinh 1 \cos x \, dx \right]$$

$$= 1 - \cosh 1 \cos 1 + i \sinh 1 \sin 1 = I_1.$$

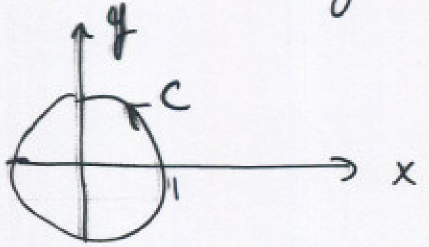
For this example, we have same result for each path. Constructing a closed contour from $O \to B$ along C_1 followed by $B \to A \to O$ along "$-C_2$", our result:

$$\oint \sin z \, dz = 0$$



The result $\oint_C f(z)\,dz = 0$, equivalent to independence of the path of integration will turn out to be general for $f(z)$ analytic on and within the contour C.

2. Two examples using closed contour, counterclockwise around unit circle centered at origin.



On this contour,

$$z = e^{i\theta} \qquad dz = ie^{i\theta}\,d\theta$$

a) $f(z) = z^*$ On C, $f(z) = e^{-i\theta}$.

$$\oint_C f(z)\,dz = \int_0^{2\pi} (e^{-i\theta})(ie^{i\theta}\,d\theta) = 2\pi i$$

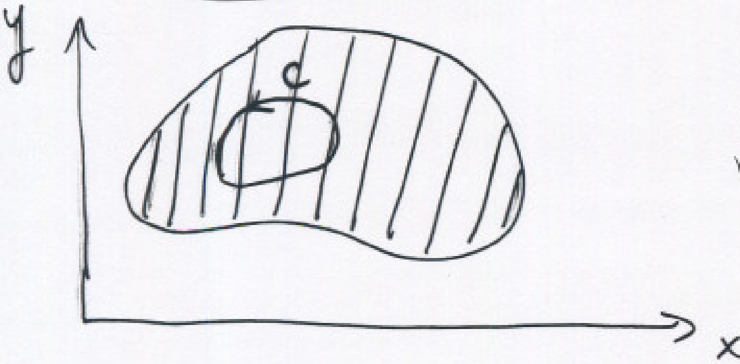
b) $f(z) = \frac{1}{z}$ On C, $f(z) = e^{-i\theta}$ also.

$$\therefore \oint f(z)\,dz = 2\pi i$$

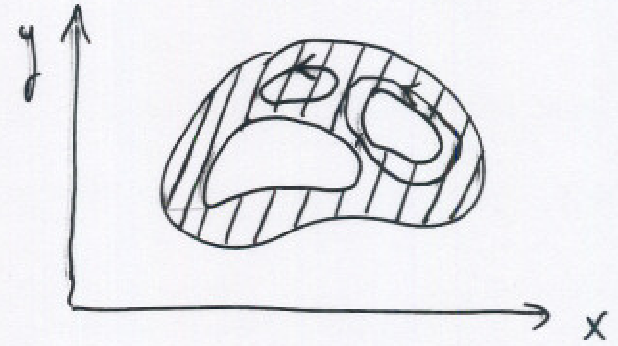
Already saw z^* not analytic. $f(z) = \frac{1}{z}$ not analytic at $z=0$.

A region R in the complex plane is called simply connected if all possible closed curves within it contain only points belonging to R.

no holes Otherwise, multiply connected.



vs



Cauchy's Theorem (Cauchy-Goursat thm)

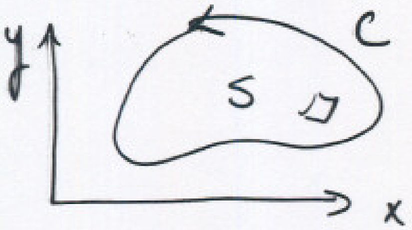
If a function $f(z)$ is analytic on a closed curve C and within the whole region R enclosed by C (simply connected), then $\oint_C f(z)\, dz = 0$.

[Original form of thm needs assumption not only that derivative of $f(z)$ exists but that it's continuous throughout the region. Goursat developed proof that does not need continuous derivative. Not of practical importance to us since we'll see that analytic fns have derivatives of all orders, implying continuity [$f'(z)$ cont. since $f''(z)$ exists...]

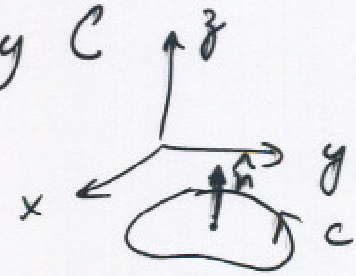
Goursat's proof consists of dividing region bdd by C into subregions $\oint_C \le \sum_i \oint_{C_i} \to \epsilon \to 0$, using results on upper limits of subintegrals.

Including assumption of continuity of $f'(z)$, use a proof based on Stoke's theorem.

Stoke's theorem for the contour integral of a vector function in a 2-d plane:



S enclosed by C



$\hat{n} = \hat{k}$

$$\vec{A} = A_x \hat{\imath} + A_y \hat{\jmath}$$

$$\oint_C \underbrace{\vec{A}\cdot\vec{dl}}_{} = \int_S \underbrace{(\vec{\nabla}\times\vec{A})}_{}\cdot\hat{n}\,\underbrace{dS}_{} \qquad dS = dx\,dy$$

$$\hat{n} = \hat{k}$$

$$\vec{dl} = dx\,\hat{\imath} + dy\,\hat{\jmath} \qquad (\vec{\nabla}\times\vec{A})_k = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

Apply this to

$$\oint_C f(z)\,dz = \oint_C \underbrace{(u\,dx - v\,dy)}_{} + i\oint_C \underbrace{(u\,dy + v\,dx)}_{\text{Identify}}$$

Identify $\vec{F} = u\hat{\imath} - v\hat{\jmath}$; then $\vec{F}\cdot\vec{dl} = u\,dx - v\,dy$

$\vec{G} = v\hat{\imath} + u\hat{\jmath}$; then $\vec{G}\cdot\vec{dl} = u\,dy + v\,dx$

$$\oint_C f(z)\,dz = \oint_C \vec{F}\cdot\vec{dl} + i\oint_C \vec{G}\cdot\vec{dl}$$

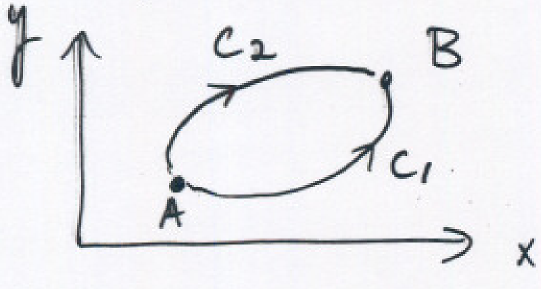
$$= \int_S (\vec{\nabla}\times\vec{F})\cdot d\vec{s} + i\int_S (\vec{\nabla}\times\vec{G})\cdot d\vec{s}$$

$$= \int_S dx\,dy\,\underbrace{\left[\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\right]}_{=0 \text{ by}} + i\int_S dx\,dy\,\underbrace{\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right]}_{=0}$$

Cauchy-Riemann eqns.

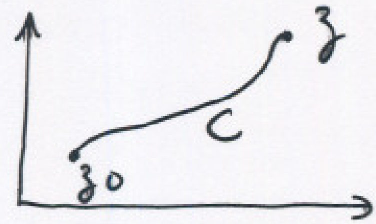
$= 0$

Since Cauchy's thm gives $\oint_C f(z)\,dz = 0$ for a closed contour, it follows that a contour integral depends only on endpoints — path independence



$$\int_{C_1} = \int_{C_2}$$

Armed with Cauchy's theorem, return to the fundamental theorem of calculus.



If $f(z)$ is analytic in the region containing C, by Cauchy's theorem we know that

$$F(z) = \int_{z_0}^{z} dz'\, f(z')$$

defines a unique function, independent of the path from z_0 to z.

note: typo Wyld 10.5.23

Then

$$F(z+\Delta z) = \int_{z_0}^{z+\Delta z} dz'\, f(z')$$

$$= \underbrace{\int_{z_0}^{z} dz'\, f(z')}_{= F(z)} + \int_{z}^{z+\Delta z} dz'\, f(z')$$

$$\therefore \quad F(z+\Delta z) - F(z) = \int_{z}^{z+\Delta z} dz'\, f(z')$$

Write

$$f(z) = \frac{f(z)}{\Delta z} \int_{z}^{z+\Delta z} dz' = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} dz'\, f(z)$$

Then

$$\left[\frac{F(z+\Delta z) - F(z)}{\Delta z}\right] - f(z) = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} dz'\, f(z') - \frac{1}{\Delta z} \int_{z}^{z+\Delta z} dz'\, f(z)$$

$$= \frac{1}{\Delta z} \int_{z}^{z+\Delta z} dz' \left[f(z') - f(z) \right]$$

$f(z)$ is analytic. Thus it is continuous.
So, for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $|z' - z| < \delta$, then $|f(z') - f(z)| < \epsilon$.

Take $0 < |\Delta z| < \delta$.

Then $$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| < \frac{\epsilon}{\Delta z} \int_z^{z+\Delta z} |dz'| = \epsilon$$

Reexpressed $$\lim_{\Delta z \to 0} \left[\frac{F(z+\Delta z) - F(z)}{\Delta z} \right] = f(z)$$

Recognize this as derivative of $F(z)$.

$F'(z) = f(z)$ So $F(z)$ is analytic $\rightarrow$ its derivative exists and is equal to $f(z)$.

Thus the integral $F(z)$ of an analytic fn $f(z)$ is an analytic fn of its upper limit, provided the integration path is in a region where $f(z)$ is analytic.

$$\int_a^b f(z')dz' = \int_{z_0}^b (f(z')\,dz') - \int_{z_0}^a f(z')dz'$$

$$= F(b) - F(a)$$

$$\frac{dF}{dz} = f$$

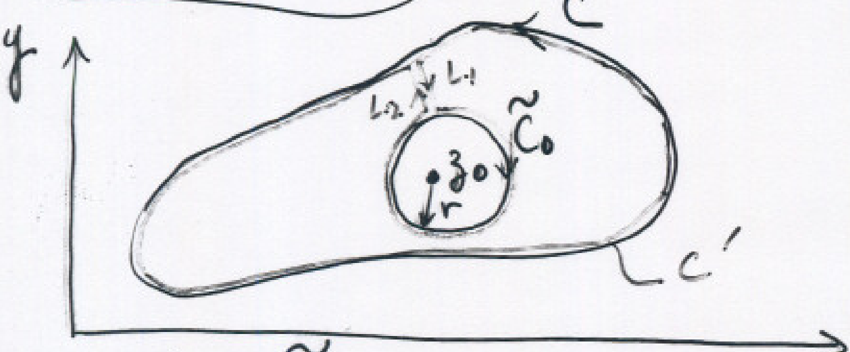
Cauchy's Integral Formula

Theorem: If $f(z)$ is analytic within and on a closed contour C (implies a simply connected region R), then

$$\frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{z - z_0} = \begin{cases} f(z_0) & \text{if } z_0 \text{ is inside } C \\ 0 & \text{if } z_0 \text{ exterior to } C \end{cases}$$

Note denominator.

Proof for z_0 interior to C:



$\tilde{C}_0$ is a circle of radius r centered at z_0

C' is the original contour C along with $\tilde{C}_0$ joined by L_1 and L_2, which are arbitrarily close to each other and oppositely directed.

C taken counterclockwise
$\tilde{C}_0$ clockwise

Consider $\displaystyle\oint_{C'} \frac{f(z)}{z - z_0}\,dz$

C' does not enclose z_0. Hence the integrand $\dfrac{f(z)}{z-z_0}$ will be analytic within and on C'.

By Cauchy's theorem:

$$\oint_{C'} \frac{f(z)}{z-z_0}\,dz = 0 = \oint_C (\quad)\,dz + \cancel{\int_{L_1} (\quad)\,dz} + \int_{\tilde{C}_0} (\quad)\,dz + \cancel{\int_{L_2} (\quad)\,dz}$$

As the line segments L_1 and L_2 approach each other $\int_{L_1} \longrightarrow -\int_{L_2}$, so they cancel

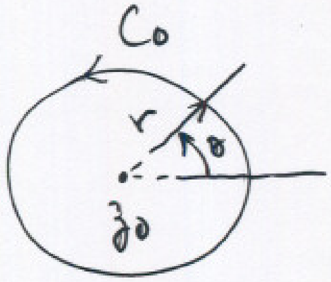
$\tilde{C}_0$ is traversed clockwise so call C_0 the same path but traversed counterclockwise

$$\oint_{\tilde{C}_0} = -\oint_{C_0}$$

$$\oint_{C'} \frac{f(z)\,dz}{z-z_0} = 0 = \oint_C \frac{f(z)\,dz}{z-z_0} - \oint_{C_0} \frac{f(z)\,dz}{z-z_0}$$

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \oint_{C_0} \frac{\overbrace{f(z)}^{= f(z)-f(z_0)+f(z_0)}\,dz}{z-z_0}$$

$$= f(z_0)\oint_{C_0} \frac{dz}{z-z_0} + \oint_{C_0} \frac{[f(z)-f(z_0)]\,dz}{z-z_0}$$



On C_0:

$z-z_0 = re^{i\theta}$ $\qquad dz = ire^{i\theta}d\theta$

$$\therefore \oint_{C_0} \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{ire^{i\theta}d\theta}{re^{i\theta}} = 2\pi i \,, \text{ for } r>0 \text{ within } C.$$

Now investigate the integral $\oint_{C_0} \frac{[f(z)-f(z_0)]\,dz}{z-z_0}$.

$f(z)$ is continuous so for any $\epsilon>0$ there exists a $\delta>0$ such that if $|z-z_0|<\delta$, then $|f(z)-f(z_0)|<\epsilon$.

If we take $r=\delta$, we have $|z-z_0|=\delta$.

$$\left|\oint_{C_0} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \le \oint_{C_0} \frac{\overbrace{|f(z)-f(z_0)|}^{\epsilon}\ \overbrace{|dz|}^{2\pi\delta}}{\underbrace{|z-z_0|}_{\delta}} = 2\pi\epsilon$$

Taking r small but nonzero, the absolute value of this integral can be made smaller than any preassigned value ϵ. So the integral $\longrightarrow 0$.

$$\therefore \boxed{\oint_C \frac{f(z)\,dz}{z-z_0} = 2\pi i\, f(z_0)} \quad \text{for } z_0 \text{ within } C.$$

If z_0 is exterior to C, then the integrand $\frac{f(z)}{z-z_0}$ is analytic within and on C

and so
$$\boxed{\oint \frac{f(z)\,dz}{z-z_0} = 0 \quad \text{for } z_0 \text{ outside } C.}$$

This proves Cauchy's integral formula. It tells us that the value of an analytic function at any point within a contour C is determined by its value on the contour.

$\longrightarrow$ analytic fns: both real and imaginary parts satisfy Laplace's eqn.

For $\nabla^2 V = 0$, given V on some boundary, can determine V within the boundary.

$$f(z) = \oint_C dz' \left[\frac{1}{2\pi i\,(z'-z)}\right] f(z')$$

This is a special case of the integral representation of a function $f(z)$. More generally,

$$f(z) = \int_C dz' \, \underbrace{K(z,z')}_{\text{Kernel}} \, g(z')$$

Derivatives of an analytic function

From Cauchy's integral formula, it follows that all derivatives of an analytic function are analytic. (not true for real fns)

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{z - z_0} \qquad \text{C's I. F.}$$

Differentiate both sides w.r.t. z_0, interchanging the order of integration and differentiation on RHS.

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^2}$$

(can be made rigorous using defn of diffn. and limiting procedure)

Repeating this process, the nth derivative wrt z_0 is —

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

Thus the derivatives to all orders exist. The kth derivative is continuous <u>because</u> the (k+1)th derivative exists.

So if a function is analytic, then its partial derivatives exist and are continuous. (Cauchy - Goursat thm)

Analytic Functions and Power Series

Briefly review (mention) some concepts for series.

A series $\sum_{n=0}^{\infty} a_n$ is convergent to a limit f if

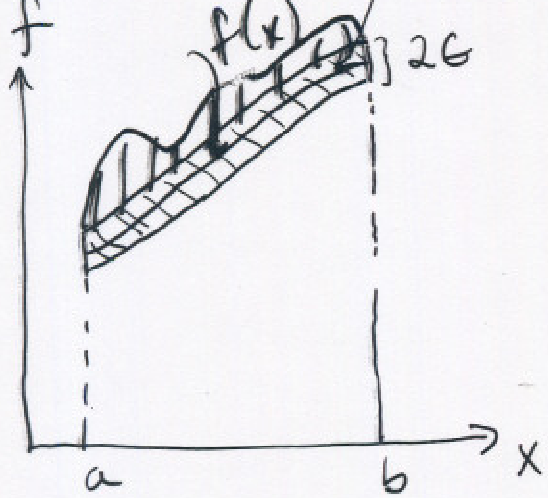
for arbitrary $\epsilon > 0$, $\exists\ N_0$ such that

$$\left| \sum_{n=0}^{N} a_n - f \right| < \epsilon \quad \text{for all } N \geq N_0.$$

Then $\sum_{n=0}^{\infty} a_n = f$.

If the equivalent series of absolute values $\sum_{n=0}^{\infty} |a_n|$ is convergent, it is called absolutely convergent.

In the case that the series elements are functions of a variable, we have the concept of uniform convergence. Consider the idea in terms ~~of~~ fns of a real variable. Say we have an infinite sequence of functions $f_1(x), f_2(x), \ldots f_n(x), \ldots$ This sequence converges to $f(x)$ if, for any x, and for arbitrary $\epsilon > 0$, there exists N_0 such that $|f(x) - f_n(x)| < \epsilon$ for $n > N_0$. If N_0 is independent of x, the sequence converges uniformly.



If the uniformly convergent sequence $S_n(x)$ is the sequence of partial sums $S_n(x) = \sum_{m=0}^{n} f_m(x)$ then the series $\sum_{n=0}^{\infty} f_n(x)$ is uniformly convergent.

This concept of uniform convergence goes over to the case of complex variables.

There are various tests for convergence. Some rely on

$$|a_N + a_{N+1} + \cdots + a_{N+p}| \leq |a_N| + |a_{N+1}| + \cdots + |a_{N+p}|$$

Since this relation uses absolute values, it is also A-OK for complex numbers.

Comparison test: If $\sum S_n$ is a convergent series of positive real terms and if $|a_n| < S_n$, then $\sum a_n$ is absolutely convergent.

Ratio test: Assuming the limit exists, let

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lambda \qquad \text{(successive terms)}$$

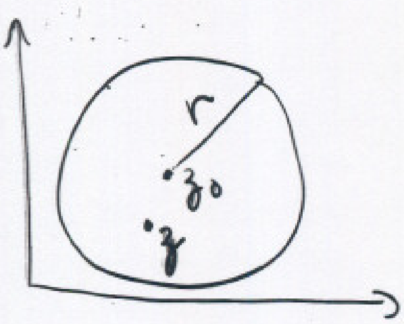
If $\lambda < 1$, $\sum a_n$ converges absolutely; if $\lambda > 1$, $\sum a_n$ diverges.

Consider the condition for convergence of a power series $\sum a_n = \sum \alpha_n (z - z_0)^n$:

– convergent for $\lim_{n\to\infty} \left| \frac{\alpha_{n+1}(z-z_0)^{n+1}}{\alpha_n (z-z_0)^n} \right|$

$$= |z - z_0| \lim_{n\to\infty} \left| \frac{\alpha_{n+1}}{\alpha_n} \right| < 1$$

Namely, for $|z - z_0| < r = \dfrac{1}{\lim_{n\to\infty} \left| \frac{\alpha_{n+1}}{\alpha_n} \right|}$ → convergent



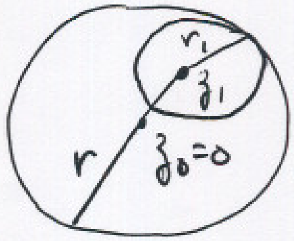
The series $f(z) = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$ converges for z within a circle of radius r about z_0.

radius of convergence

circle of convergence

Consider a power series about the point $z_0 = 0$ with radius of convergence r:

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n z^n}$$



1. It is unique.

2. Can be developed as a unique power series about any other point z_1 within the circle of convergence.

$$f(z) = \sum_{n=0}^{\infty} b_n (z - z_1)^n$$ with radius of convergence at least $r_1 = r - |z_1|$.

3. $f(z)$ is differentiable everywhere within c. of c. with derivative (at z_1)

$$f'(z_1) = \sum_{n=1}^{\infty} n\, a_n (z_1)^{n-1}$$

* Thus a function that can be represented as a power series is analytic within its radius of convergence. *

The kth derivative at z_1 is

$$f^{(k)}(z_1) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1)\, a_n z_1^{n-k}$$

We can identify a_n by evaluating this for $z_1 = 0$. Only the $n = k$ term contributes.

$$a_k = \frac{f^{(k)}(0)}{k!} \qquad \text{so}$$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

Taylor series about $z=0$

Generalizing $\displaystyle f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)}{n!} (z-z_1)^n$

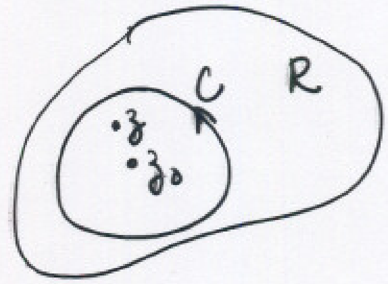
Now consider uniformly convergent series of analytic functions.

$$F(z) = \sum_{n=0}^{\infty} f_n(z) \quad \longleftarrow$$

converges uniformly in region R in z-plane

This can be integrated term by term.

$$\oint_C F(z)\,dz = \sum_{n=0}^{\infty} \oint_C dz\, f_n(z)$$



Apply this idea to consider Cauchy's integral formula for an analytic function in terms of a power series expansion about a point z_0.

C's I.F. $\displaystyle f(z) = \frac{1}{2\pi i} \oint_C dz' \frac{f(z')}{(z'-z)}$

z' is on C

$$\frac{1}{z'-z} = \frac{1}{z'-z_0-(z-z_0)} = \frac{1}{(z'-z_0)\left[1-\frac{(z-z_0)}{(z'-z_0)}\right]}$$

Since z' is on C, and z is within C, $\frac{|z-z_0|}{|z'-z_0|} < 1$.

Recall that, for the geometric series,

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{for } |z|<1. \qquad \frac{1}{1-\epsilon} = \sum_{n=0}^{\infty} \epsilon^n$$

This is the form we have:

$$\therefore \frac{1}{\left[1-\frac{(z-z_0)}{(z'-z_0)}\right]} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(z'-z_0)^n}$$

So, $$f(z) = \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\oint_C \frac{dz'\, f(z')}{(z'-z_0)^{n+1}}\right](z-z_0)^n$$

$$= \sum_{n=0}^{\infty}\left[a_n\right](z-z_0)^n$$

We recognize a_n as $\frac{1}{n!} f^{(n)}(z_0)$ (pg C48),

nth derivative evaluated at z_0

So this is just a Taylor series, which converges in circle of convergence about z_0.

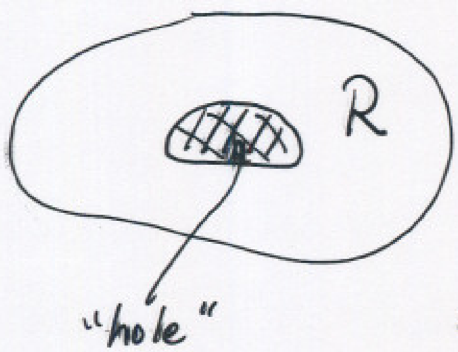
* $\boxed{\text{Thus an analytic function can be expressed as a power series.}}$ * Compare C52

Now we can generalize to the situation where the region R under consideration is not simply connected.

This will lead to another type of series expansion - Laurent series - that consists of both positive and negative powers of z.

In turn, we will use Laurent series in the evaluation of integrals via the residue theorem.

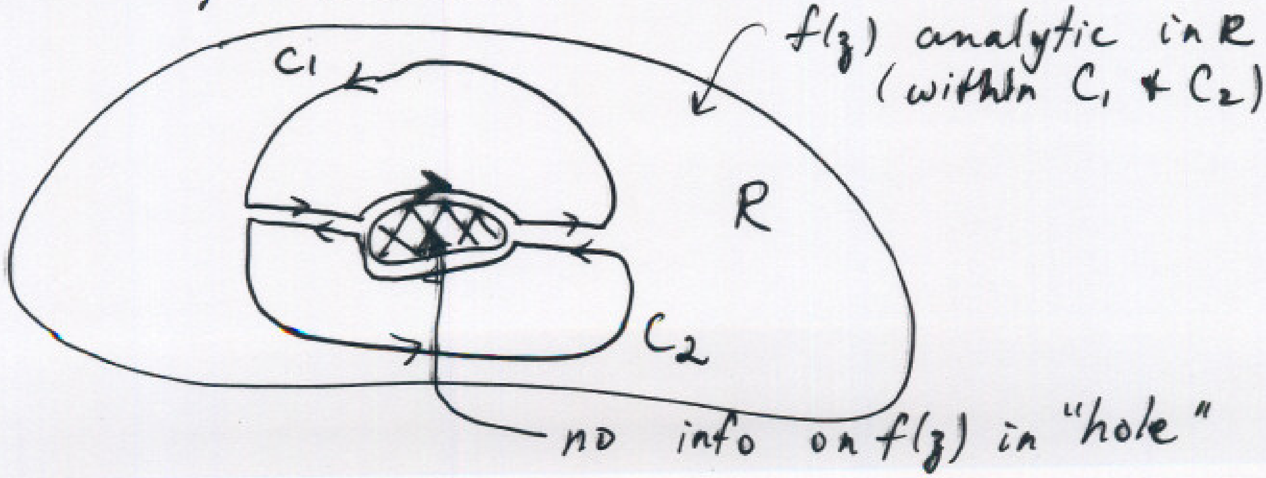
First - generalize Cauchy's theorem to a region R which is annular.



Suppose a function $f(z)$ is analytic within R.

Cauchy's theorem applies only to contours for which $f(z)$ is analytic on and within C.

Apply Cauchy's theorem for contours C_1 and C_2:



By Cauchy's theorem:

$$\oint_{C_1} f(z)\,dz = 0 \qquad \text{and} \qquad \oint_{C_2} f(z)\,dz = 0$$

Thus, $\oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz = 0$

Bringing the oppositely directed line segments of C_1 and C_2 arbitrarily close to each other, the contributions to the contour integrals from these segments will cancel in taking the sum above.

This cancellation produces an equivalent pair of contours, C_3 (ccw) and $\tilde{C}_4$ (cw):

$$\oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz = 0 =$$

$$= \oint_{C_3} f(z)\,dz + \oint_{\tilde{C}_4} f(z)\,dz$$

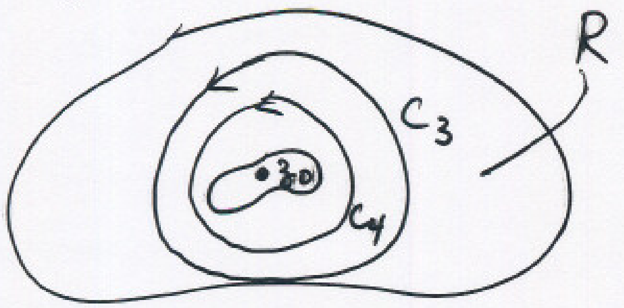
Defining ccw contour $\longrightarrow$ $C_4 \longrightarrow$

$$= \oint_{C_3} f(z)\,dz - \oint_{C_4} f(z)\,dz = 0$$

Thus, $\oint_{C_3} f(z)\,dz = \oint_{C_4} f(z)\,dz$ where C_3 and C_4 are any pair of contours lying fully within the annular region of analyticity R and circling the "hole".

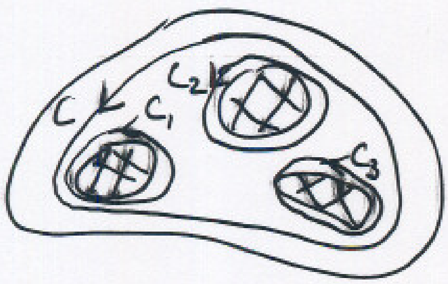
At this point, we don't know the value of such a contour integral $\oint_C f(z)dz$ circling the "hole".

In the case of an annular region, it will sometimes be useful to choose the contours C_3 and C_4 as concentric circles about a point z_0 located in the "hole".



This will sometimes make evaluation of $\oint f(z)dz$ feasible.

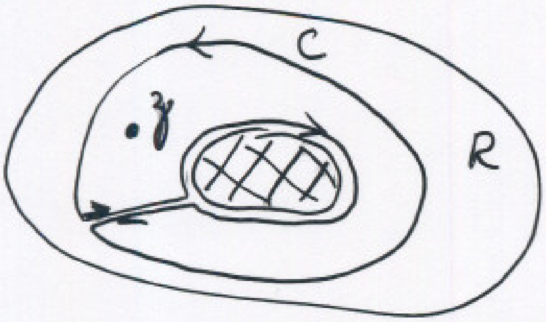
Generally for a more complicated multiply connected region



$$\oint_C f(z)\,dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \oint_{C_3} f(z)\,dz + \cdots$$

Now generalize Cauchy's integral formula to the case of an annular region of analyticity.

Assume $f(z)$ is analytic in an annular region R. Apply Cauchy's integral formula to a contour C, as shown, that lies entirely within R and encloses only points in R.



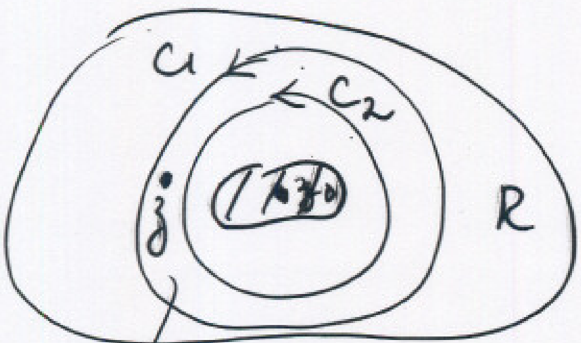
z is a point within C.

For C, Cauchy's i.f. says:

$$f(z) = \frac{1}{2\pi i} \oint_C dz' \frac{f(z')}{(z'-z)}$$

"the difference of the integrals about"

But we've just seen that C is equivalent to a pair of contours lying within R and circling the hole. Choose them to be concentric circles centred on some point z_0 in the hole.



z is lying between C_1 and C_2

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{dz'\, f(z')}{(z'-z)} = \frac{1}{2\pi i}\oint_{C_1} \frac{dz'\, f(z')}{(z'-z)} - \frac{1}{2\pi i}\oint_{C_2} \frac{dz'\, f(z')}{(z'-z)}$$

This is the generalization of Cauchy's integral formula for this annular region of analyticity.

Now deal separately with the contour integrals about C_1 and C_2 in the same way we generated the Taylor series.

C_1: z' lies on C_1 and z lies within C_1. So $|z'-z_0| > |z-z_0|$.

Express $\dfrac{1}{z'-z}$ as $\dfrac{1}{z'-z_0-(z-z_0)} = \dfrac{1}{(z'-z_0)}\dfrac{1}{\left[1-\underbrace{\left(\frac{z-z_0}{z'-z_0}\right)}_{<1}\right]}$

(geometric series)

$$= \frac{1}{(z'-z_0)}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n$$

C_2: z' lies on C_2 and z is exterior to C_2.

So $|z'-z_0| < |z-z_0|$.

$$\frac{1}{z'-z} = \frac{1}{z'-z_0-(z-z_0)} = \frac{-1}{(z-z_0)}\left[\frac{1}{1-\left(\frac{z'-z_0}{z-z_0}\right)}\right] = \frac{-1}{(z-z_0)}\sum_{n=0}^{\infty}\left(\frac{z'-z_0}{z-z_0}\right)^n$$

Insert these expansions into $*$ for $f(z)$:

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{C_1} \frac{dz' \, f(z')}{(z'-z_0)^{n+1}} (z-z_0)^n$$

$$+ \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{C_2} dz' \, f(z') \, (z'-z_0)^n \, (z-z_0)^{-n-1}$$

In the second term, change the range of the sum:

$$\sum_{\underset{\uparrow}{n=1}}^{\infty} (z'-z_0)^{n-1} \, (z-z_0)^{-n}$$

Now change the range again to go over negative integers:

$$\sum_{n=-\infty}^{-1} \frac{1}{(z'-z_0)^{n+1}} (z-z_0)^n$$

principal part

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint \frac{dz' \, f(z')}{(z'-z_0)^{n+1}} (z-z_0)^n + \sum_{n=-\infty}^{-1} \frac{1}{2\pi i} \oint \frac{dz' \, f(z')}{(z'-z_0)^{n+1}} (z-z_0)^n$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi i} \oint dz' \frac{f(z')}{(z'-z_0)^{n+1}} \right] (z-z_0)^n = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n$$

Laurent series.

Any terms with negative powers in the Laurent series of a function are symptomatic of the singularities of the function within the contour — which means within the "hole" region since the function is analytic in the annular region.

If $f(z)$ is analytic in the neighbourhood of a point $z=a$ but not at $z=a$, then a is an isolated singularity.

En route to characterizing the singularities of a function, we can first characterize zeroes.

$z=a$ is a zero of a function f if $f(a)=0$. If f is analytic at a, we can expand f in a Taylor series within the radius of convergence $|z-a|<r$ as —

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

If a is a zero of f, then $c_0 = 0$. If $c_1 \neq 0$, then a is a simple zero of f. In general, if $c_0, c_1, \ldots, c_{n-1}$ are all zero but $c_n \neq 0$, then a is a zero of order n.

The order of a zero can be found by calculating $\lim_{z \to a} \frac{f(z)}{(z-a)^n}$ for $n=1,2,3\ldots$. The order is

the lowest value of n for which this limit will not vanish. (eg $f(z)=(z-a)^3 \longrightarrow$ order 3)

Back to isolated singularities. These can be characterized in various ways.

1. $|f(z)| \to \infty$ as $z \to a$

These singularities are called poles. Since the singularity is isolated, there is a Laurent series

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-a)^n$$

If this series terminates at some value $n = -m$, then $f(z)$ has a pole of order m at $z=a$. A fn with well separated poles as its only singularities is called meromorphic.

$$f(z) = \sum_{n=-m}^{+\infty} a_n (z-a)^n$$

order $m=1$ simple pole

Can test for the order of the pole by looking at

$$\lim_{z \to a} (z-a)^m f(z)$$

The order of the pole at $z=a$ is the lowest integer m for which this limit exists.

[Note that if $f(z)$ has a simple zero at $z=a$, then $\frac{1}{f(z)}$ has a simple pole at $z=a$.]

2. $f(z)$ is bounded as $z \to a$.

This is called a removable singularity. In practice, these are functions that are given by a formula that fails at $z=a$ but for which the limit

$$\lim_{z\to a} f(z) \quad \text{exists.}$$

Example
$$f(z) = \frac{\sin z}{z} = \begin{cases} \dfrac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

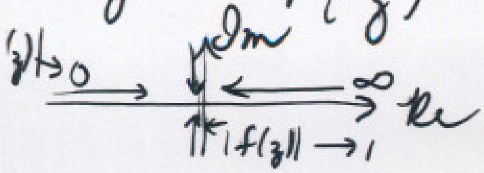
$$\lim_{z\to 0} \frac{\sin z}{z} \longrightarrow 1$$

Defined in this extended way the new function is analytic everywhere

3. For an essential singularity, the Laurent series must have an infinite principal part. This is characteristic of an essential singularity but an infinite number of negative terms does not actually imply an essential singularity.

$f(z)$ must be such that it oscillates wildly as you approach the singularity $\to$ not bounded but not $\to \infty$ either. Classic example:

$$f(z) = \exp\left(\frac{1}{z}\right) = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots$$



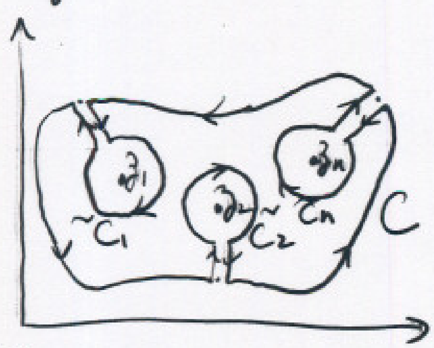
Residue Theory

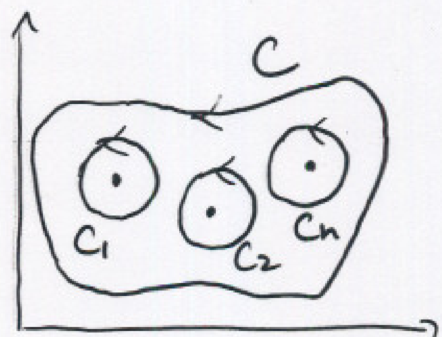
Nothing new here – just apply some of our results.

Consider a function $f(z)$ that is analytic in some region except at a finite number, n, of isolated singular points, $z_1 \rightarrow z_n$.

Residue theorem: The integral of $f(z)$ around a closed contour C containing a finite number n of singular points of $f(z)$ equals the sum of n integrals about n circles, each enclosing only one of the singular points.

Follows immediately from our generalization of Cauchy's theorem to a multiply connected region (pg C56).





C_i ccw

$\tilde{C}$ cw

C ccw

typo Wyld (11.2.1)

$$\oint_C f(z)\,dz = \sum_{i=1}^{n} \oint_{C_i} dz\, f(z) \equiv 2\pi i \sum_{i=1}^{n} \operatorname{Res} f(z_i)$$

where $\operatorname{Res} f(z_i) = \frac{1}{2\pi i} \oint_{C_i} dz\, f(z)$ is the residue of $f(z)$ at $z = z_i$.

In $\operatorname{Res} f(z_i)$, C_i is a ccw oriented contour enclosing at most one singularity. If $f(z)$ is analytic throughout the region enclosed by C_i, then $\operatorname{Res} f(z_i) = 0$ by Cauchy's theorem.

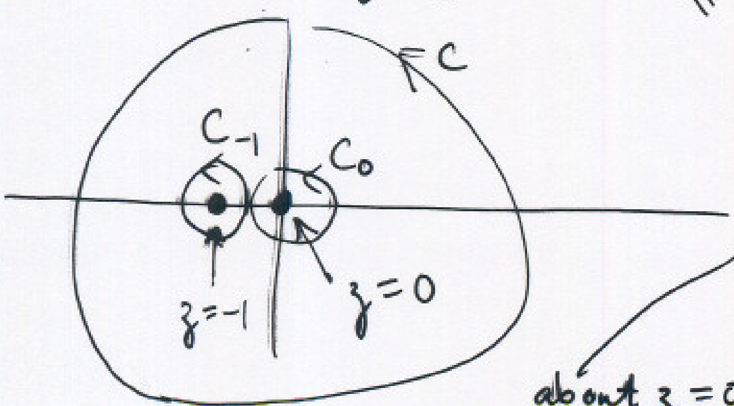
[However, $\operatorname{Res} f(z_i) = 0$ does not imply that $f(z_i)$ is analytic at $z = z_i$.]

There are a variety of ways to calculate the residue of a function at a point.

① Directly use Cauchy's integral formula or the formula for the nth derivative of an analytic function as derived from Cauchy's thm. (pg C 48)

Example: Evaluate the contour integral

$$I = \oint_C \frac{(3z+2)}{z(z+1)^3}\,dz \quad \text{where } C \text{ is the circle } |z| = 3.$$



$$= \oint_{C_0} \frac{\left[(3z+2)/(z+1)^3\right]}{z}\,dz \quad \text{about } z=0 \text{ with } r<1$$

$$+ \oint_{C_{-1}} \frac{\left[(3z+2)/z\right]}{(z+1)^3}\,dz \quad \text{about } z=-1 \text{ with } r<1$$

$$= 2\pi i \left\{ \operatorname{Res} f(z=0) + \operatorname{Res} f(z=-1) \right\}$$

$$I = \oint_{C_0} \frac{f_0(z)}{z}\,dz + \oint_{C_{-1}} \frac{f_{-1}(z)}{(z+1)^3}$$

$f_0(z)$ is analytic on and within C_0. This integral is form dealt with by Cauchy's integral formula

$f_{-1}(z)$ is analytic for C_{-1}. This is the form arising in the expression for the second derivative of f_{-1}

$$\oint_{C_0} = 2\pi i \underbrace{f_0(z=0)}_{=2}$$

$$\oint_{C_{-1}} = \frac{2\pi i}{2!} \underbrace{f_{-1}^{(2)}(z=-1)}_{=-4}$$

$$I = 2\pi i \left(2 - \frac{4}{2!}\right) = 0$$

Other methods to evaluate the residue of a function rely on making a Laurent expansion.

The residue of $f(z)$ at $z = z_i$ is the coefficient a_{-1} of the Laurent expansion of f about $z = z_i$.

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n$$

Revert to separating the positive & negative powers.

$$= \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

$$\boxed{b_1 = a_{-1}}$$

In terms of this expansion:

$$\oint_{C_i} f(z)\,dz = \sum_{n=0}^{\infty} a_n \oint_{C_i} dz\,\underbrace{(z-z_i)^n} + \sum_{n=1}^{\infty} b_n \oint_{C_i} \frac{dz}{(z-z_i)^n}$$

For +ve n, this integrand is analytic so, by Cauchy's theorem, each integral in the sum is zero

$$= \underbrace{b_1 \oint_{C_i} \frac{dz}{(z-z_i)}}_{=2\pi i} + \sum_{n=2}^{\infty} b_n \underbrace{\oint_{C_i} \frac{dz}{(z-z_i)^n}}$$

by Cauchy's integral formula

Integral is of form $g^{(n-1)}(z_i)$ with $g(z)=1$.

$g' = 0$; $g'' = 0$; ...

Recall nth derivative of fn $g(z)$:

$$g^{(m)}(z_i) = \frac{m!}{2\pi i} \oint_{C_i} \frac{g(z)\,dz}{(z-z_i)^{m+1}}$$

$\therefore$ All these integrals vanish.

$$\therefore \oint_{C_i} f(z)\,dz = 2\pi i\, b_1 = 2\pi i\, a_{-1}$$

$$\text{Res}\, f(z_i) = \frac{1}{2\pi i} \oint_{C_i} f(z)\,dz = a_{-1}$$

In certain cases, there are simple formulae for pulling out this coefficient of the Laurent series and, hence, the residue

(2) Say $f(z)$ has a simple pole (order 1) at $z = \alpha$. Then the Laurent series has the form

$$f(z) = \frac{a_{-1}}{(z-\alpha)} + a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots$$

In this case,

$$\text{Res}\, f(\alpha) = a_{-1} = \lim_{z\to\alpha} (z-\alpha) f(z)$$

All other term vanish due to factor of $(z-\alpha)^p$ with $p \geq 1$.

(3) For a pole of order m, the residue is

$$\text{Res}\, f(\alpha) = \lim_{z\to\alpha} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-\alpha)^m f(z) \right]$$

Proof:
The Laurent series about $z = \alpha$ is, in this case,

$$f(z) = \sum_{n=-m}^{\infty} a_n (z-\alpha)^n$$

We want to evaluate

$$\lim_{z\to\alpha} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[\sum_{n=-m}^{\infty} a_n (z-\alpha)^{m+n} \right]$$

For powers $(m+n) < (m-1)$, the number of derivatives, the derivative vanishes.

Namely, only $n \geq -1$ contribute a nonvanishing derivative.

$$\lim = \lim_{z \to \alpha} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \sum_{n=-1}^{\infty} a_n (z-\alpha)^{m+n}$$

$$= \lim_{z \to \alpha} \frac{1}{(m-1)!} \sum_{n=-1}^{\infty} a_n (m+n)(m+n-1) \cdots (n+2)(z-\alpha)^{n+1}$$

In the limit $z \to \alpha$, only the term $n=-1$ contributes a nonzero value.

$$\lim = \lim_{z \to \alpha} \frac{1}{(m-1)!} (m-1)(m-2) \cdots (1) a_{-1} = a_{-1}$$

Thus the result is proven.

(4) Consider $f(z)$ of the form $f(z) = \dfrac{p(z)}{q(z)}$

where $p(z)$ is analytic at and near $z=\alpha$ and $q(z)$ has a simple zero at $z=\alpha$. In this case $f(z)$ has a simple pole at $z=\alpha$. (Check that.)

In this case $\operatorname{Res} f(\alpha) = \dfrac{p(\alpha)}{q'(\alpha)}$.

This is established by expressing $q(z)$ as a Taylor series –

$$q(z) = \sum_{n=1}^{\infty} c_n (z-\alpha)^n$$ where

$$c_n = \frac{1}{n!} \frac{d^n q}{dz^n}\bigg|_{z=\alpha}$$

$c_0 = 0$ since $q(z)$ has a simple zero at $z = \alpha$

– and using recipe #2 for $f(z)$ with a simple pole –

$$\lim_{z \to \alpha} (z-\alpha) f(z) = \lim_{z \to \alpha} \frac{(z-\alpha)\, p(z)}{\sum_{n=1}^{\infty} c_n (z-\alpha)^n}$$

$$= \lim_{z \to \alpha} \frac{p(z)}{\sum_{n=1}^{\infty} c_n (z-\alpha)^{n-1}} = \lim_{z \to \alpha} \frac{p(z)}{[c_1 + \underbrace{c_2 (z-\alpha) + \cdots}_{\text{these terms vanish for } z=\alpha}]}$$

$$= \frac{p(\alpha)}{c_1} \quad \text{where } c_1 = \frac{dq}{dz}\bigg|_{z=\alpha} \equiv q'(\alpha)$$

$$= \frac{p(\alpha)}{q'(\alpha)}$$

Thus the result is proven.

⑤ Actually expand $f(z)$ in a Laurent series and identify the coeffient a_{-1}.

This method is often effective when $f(z)$ can be expressed as the product of functions with known Laurent series.

With the techniques of residue theory, we can evaluate various definite real and complex integrals.

Consider several categories.

1. Angular integrals

Consider integrals with respect to an angular variable θ with the integrand a rational function of $\cos\theta$ and $\sin\theta$.

(form is $f(\cos\theta, \sin\theta) = \dfrac{a_0 + a_1\cos\theta + a_2\sin\theta + a_3\cos^2\theta + \cdots}{b_0 + b_1\cos\theta + b_2\sin\theta + b_3\cos^2\theta + \cdots}$)

The standard technique is to transform $\int d\theta\, f(\cos\theta, \sin\theta)$ to a complex integral $\oint dz\, R(z)$ using $z = e^{i\theta}$.

Change of variables:

$$z = e^{i\theta}$$

$$dz = ie^{i\theta}\,d\theta$$

$$\therefore \boxed{d\theta = \frac{dz}{iz}}$$

$$\boxed{\begin{aligned} \cos\theta &= \frac{1}{2}\left(z + \frac{1}{z}\right) \\ \sin\theta &= \frac{1}{2i}\left(z - \frac{1}{z}\right) \end{aligned}}$$

In the case that the integral in θ is over $0 \to 2\pi$ then the (closed) contour in z becomes the unit circle.

Then the residue theorem can be used to evaluate the integral.

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta = -i \oint_{\substack{\text{unit} \\ \text{circle}}} f\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right) \frac{dz}{z}$$

Often can put an integral into this closed contour form even if the range of integration of θ is not $0 \to 2\pi$. For instance:

a) for f an even function of $\sin\theta$:

$$\boxed{\int_0^{\pi} f(\sin^2\theta)\,d\theta = \frac{1}{2}\int_{-\pi}^{\pi} f(\sin^2\theta)\,d\theta}$$

b) For f a function of $\cos\theta$, use $\cos\theta = \cos(2\pi-\theta)$ to yield

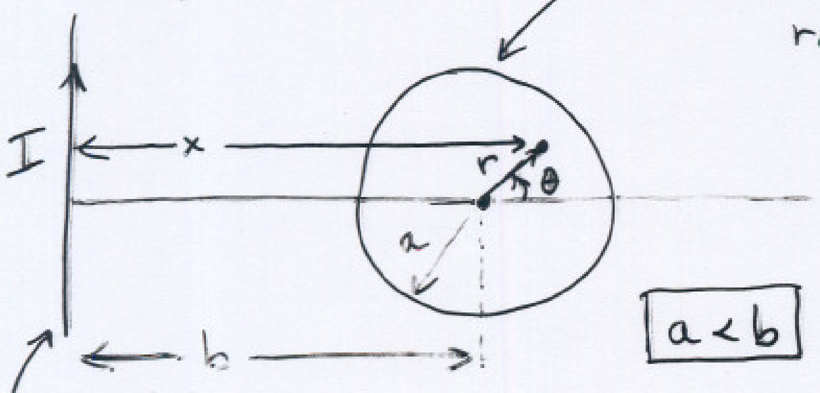
$$I = \int_0^{\pi} d\theta\, f(\cos\theta) = \int_0^{\pi} d\theta\, f(\cos\underbrace{(2\pi-\theta)}_{\theta'})$$

$$= \int_{2\pi}^{\pi} (-d\theta')\, f(\cos\theta') = \int_{\pi}^{2\pi} d\theta'\, f(\cos\theta') = I$$

$$\therefore \quad 2I = \int_0^{2\pi} d\theta\, f(\cos\theta)$$

$$\boxed{\int_0^{\pi} d\theta\, f(\cos\theta) = \frac{1}{2}\int_0^{2\pi} d\theta\, f(\cos\theta)}$$

Example: circular loop of wire, resistance R, radius a



$x = b + r\cos\theta$

$\boxed{a < b}$

long wire, with current I increasing at rate $\alpha = \frac{dI}{dt}$.

Find the current induced in the loop.

$$I_{loop} = \frac{\mathcal{E}}{R} \leftarrow \text{emf induced in loop due to changing magnetic flux}$$

$$\mathcal{E} = \left|\frac{d\Phi}{dt}\right| = \left|\frac{d}{dt}\int \vec{B}\cdot d\vec{A}\right| \qquad B = \frac{\mu_0 I}{2\pi x}\ ; \ dA = r\,dr\,d\theta$$

The magnetic flux through the loop is

$$\Phi = \int \vec{B} \cdot d\vec{A} = \int_0^a r\,dr \int_0^{2\pi} d\theta \, \frac{\mu_0 I}{2\pi (b + r\cos\theta)}$$

So we have an angular integral of the form

$$I_\theta = \int_0^{2\pi} \frac{d\theta}{(b + r\cos\theta)} \qquad \text{where } r \text{ ranges from } 0 \to a$$

$$\xrightarrow{z = e^{i\theta}} \; = \oint_{\substack{\text{unit}\\ \text{circle}}} \frac{dz}{iz} \frac{1}{\left[b + \frac{r}{2}\left(z + \frac{1}{z}\right)\right]} = -\frac{2i}{r} \underbrace{\oint \frac{dz}{\left[z^2 + 1 + 2\frac{b}{r} z\right]}}_{\equiv \oint dz\, f(z)}$$

[redacted]

The poles of the integrand are at the roots of the denominator.

$$z_\pm = -\frac{b}{r} \pm \sqrt{\left(\frac{b}{r}\right)^2 - 1}$$

Note that $z_+ z_- = 1$, so only one root will lie inside the unit circle. In this case, [redacted] $z_- < -1$ so only z_+ is within the circle.

$$I_\theta = -\frac{2i}{r} (2\pi i) \operatorname{Res} f(z_+)$$

z_+ is a simple pole so we can use recipe ② to evaluate the residue.

$$\text{Res}\, f(z_+) = \lim_{z\to z_+} (z-z_+)\, f(z) = \lim_{z\to z_+} \frac{1}{(z-z_-)} = \frac{1}{z_+-z_-}$$

$$= \frac{1}{2\sqrt{\left(\frac{b}{r}\right)^2-1}}$$

Thus $\quad I_\theta = \dfrac{2\pi}{\sqrt{b^2-r^2}}$

The rest of the calculation proceeds as usual.

$$\Phi = \mu_0 I \int_0^a \frac{r\,dr}{\sqrt{b^2-r^2}} \underset{\uparrow\, y=b^2-r^2}{=} -\mu_0 I\left[\sqrt{b^2-a^2}-b\right]$$

$$I_{loop} = \frac{\mathcal{E}}{R} = \frac{1}{R}\left|\frac{d\Phi}{dt}\right| = \frac{\mu_0 \alpha}{R}\left[b-\sqrt{b^2-a^2}\right]$$

2. Rational functions $f(x)$ (ratio of two polynomials) with no poles on the real axis integrated from $-\infty \longrightarrow +\infty$.

Under certain conditions integrals of the form $\int_{-\infty}^{+\infty} dx\, f(x)$ may be converted to equivalent contour integral $\oint f(z)\, dz$, and the residue theorem applied.

[Note: for $f(x)$ even $\left(f(-x) = f(x)\right)$, $\int_0^{\infty} dx\, f(x) = \frac{1}{2}\int_{-\infty}^{+\infty} dx\, f(x)$, so the process applies.]

The integral $\int_{-\infty}^{\infty} dx\, f(x)$ will exist if $f(z) \to 0$ as $|z| \to \infty$ at least as fast as $\frac{1}{|z|^2}$. Expressing $f(x)$ as the ratio of two polynomials, the integral exists if the degree of the denominator is at least 2 units higher than that of the numerator.

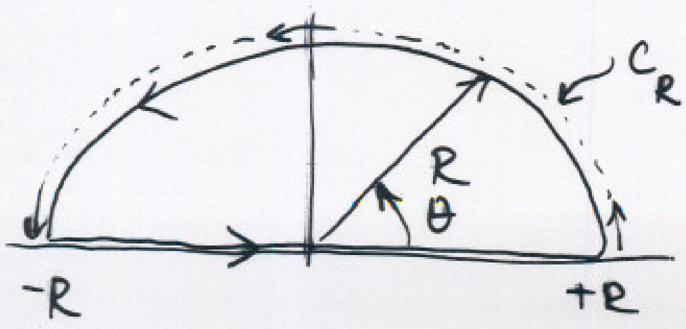
$$f(x) = \frac{\sum_{n=0}^{N} a_n x^n}{\sum_{m=0}^{M} b_m x^m} \qquad \boxed{M \geq N+2}$$

Take the contour in the z-plane as the real axis plus the semicircle $R \to \infty$ in the upper half plane (we "close the contour at ∞").

$$\int_{-\infty}^{+\infty} f(x)dx \longrightarrow \oint f(z)\, dz$$

$$= \lim_{R \to \infty} \int_{-R}^{+R} f(x)\, dx) + \lim_{R \to \infty} \underset{C_R}{\int_0^{\pi}} f(\underbrace{Re^{i\theta}}_{z}) \underbrace{iRe^{i\theta} d\theta}_{dz}$$

$$= 2\pi i \sum (\text{Residues in upper half plane})$$



This relies on the integral on the semicircle C_R vanishing as $R \to \infty$.

Using $\left| \int_{C_R} f(z)\, dz \right| \leq (\max |f(z)|) \cdot [\text{length of path}]$

and $f(z) \sim \dfrac{\text{const}}{z^2}$ $\qquad |f(z)| \sim \dfrac{|\text{const}|}{R^2}$ on C_R,

we have $\left| \int_{C_R} f(z)\, dz \right| \leq \int_{C_R} \dfrac{|\text{const}|\, R\, d\theta}{R^2} \sim \dfrac{\pi\, |\text{const}|}{R}$.

$$\therefore \lim_{R \to \infty} \int_{C_R} f(z)\, dz = 0$$

So the "contour at ∞" allows us to close the contour and apply the residue theorem without contributing itself.

Example: $\quad I = \displaystyle\int_{-\infty}^{+\infty} dx \frac{1}{(x^2+a^2)^2} \qquad a$, real

$$\longrightarrow \oint dz \frac{1}{(z^2+a^2)^2} = \oint dz \frac{1}{[(z+ia)(z-ia)]^2}$$

poles of order 2 at $z = \pm ia$

$= 2\pi i \sum$ Residues in upper half plane

$= 2\pi i$ Residue $f(z = +ia) = 2\pi i \displaystyle\lim_{z \to ia} \frac{d}{dz} (z-ia)^2 \frac{1}{(z^2+a^2)^2}$

$$= 2\pi i \lim_{z \to ia} \frac{d}{dz} \left[\frac{1}{(z+ia)^2} \right] = 2\pi i \lim_{z \to ia} \left[\frac{-2}{(z+ia)^3} \right] = \frac{\pi}{2a^3} = I$$

(N.B. Equally well close in lower half plane: cw vs ccw - sign)

3. Closing the contour with a rectangle

For certain functions, closing the contour with the infinite semicircle is not appropriate

- contour on $C_{R\to\infty}$ does not vanish
- infinite number of poles enclosed

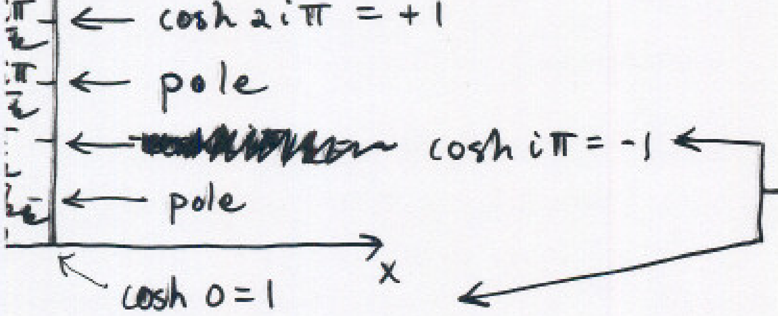
For integrands containing hyperbolic functions or e^{ax}, with a real, a rectangular contour may work.

Example to illustrate this technique:

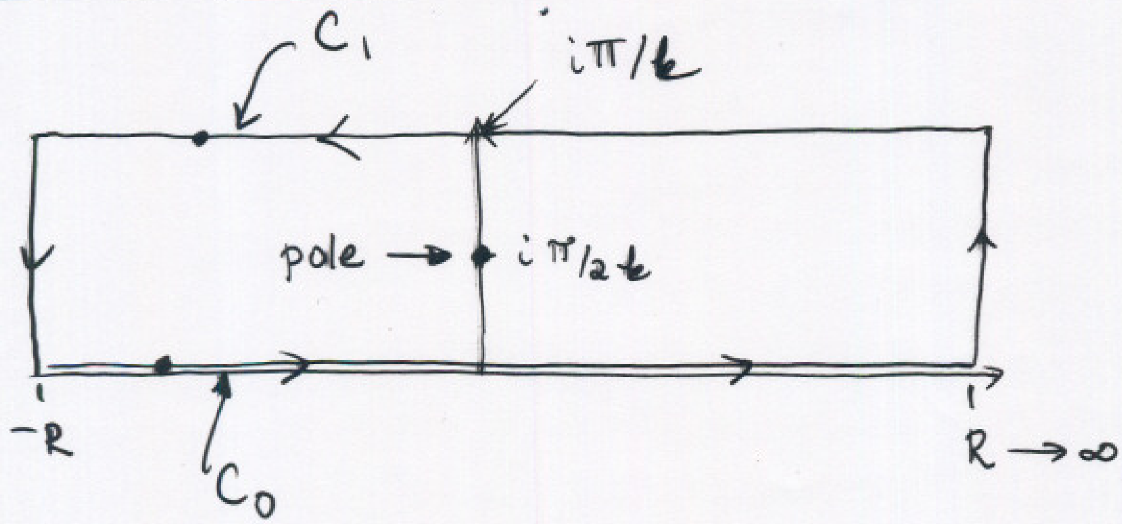
Consider $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{\cosh kx}$

$\cosh z$ has zeroes at odd multiples of $\frac{i\pi}{2}$.

$\therefore \dfrac{1}{\cosh kz}$ has poles at $z = \dfrac{i(2n+1)\pi}{2k}$, $n = 0, \pm1, \pm2\ldots$



Value of function at $z=0$ is constant factor relative to $z = \dfrac{i\pi}{k}$



Compare the integrals along C_0 and C_1:

$$\int_{C_0} = \int_{-\infty}^{+\infty} dx \frac{1}{\cosh kx} \qquad (z = x)$$

$$\int_{C_1} = \int_{+\infty}^{-\infty} dz \frac{1}{\cosh kz} \Bigg|_{z = x + i\frac{\pi}{k}} = \int_{-\infty}^{+\infty} (-dx) \frac{1}{(-\cosh kx)} = \int_{C_0}$$

$$\cosh k(x + i\pi/k) = \frac{1}{2}\left[e^{kx} e^{i\pi} + e^{-kx} e^{-i\pi}\right]$$

$$= -\cosh kx$$

Here we find the integral along C_1 is equal to that along C_0. In general, try to find a path such that the integral along the top of the rectangle is equal to a constant multiple of the integral along the real axis.

Now check the sides:

$$= \Bigg| \int_{\text{side at } x=R} \frac{dz}{\cosh kz} \Bigg| = \Bigg| i \int_0^{\pi/k} dy \frac{2}{\left[e^{kR} e^{iky} + e^{-kR} e^{-iky}\right]} \Bigg|$$

$$I_R = \frac{1}{e^{kR}} \left| \int_0^{\pi/k} \frac{dy \quad 2}{\left[e^{iky} + e^{-2kR} e^{-iky}\right]} \right| \leq \frac{1}{e^{kR}} \left| \begin{matrix}\text{max of} \\ \text{integrand}\end{matrix} \right| \frac{\pi}{k}$$

$$|\text{integrand}| = \frac{2}{\sqrt{1 + e^{-2kR} \cos 2ky + e^{-4kR}}}$$

$$\therefore \lim_{R \to \infty} I_R \sim \lim_{R \to \infty} \frac{\text{const}}{e^{kR}} = 0$$

Likewise the other side at $-R$ gives no contribution in the limit $R \to \infty$.

$$\oint_{\text{Rectangle}} = \int_{C_0} + \int_{C_1} = 2 \int_{C_0} = 2\pi i \operatorname{Res} f\left(z = \frac{i\pi}{2k}\right)$$

$$= 2 \int_{-\infty}^{\infty} \frac{dx}{\cosh kx}$$

Use recipe 4 (pg C69) to evaluate the residue.

$$f(z) = \frac{1}{\cosh kz} = \frac{p(z)}{q(z)} \qquad \begin{matrix} p = 1 \\ q = \cosh kz \end{matrix}$$

$$\operatorname{Res} f(\alpha) = \frac{p(\alpha)}{q'(\alpha)} \qquad \operatorname{Res} f\left(z = \frac{i\pi}{2k}\right) = \frac{1}{k \sinh(i\pi/2)} = \frac{1}{ik}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{\cosh kx} = \frac{\pi}{k}}$$

4. Integrals of the form $I = \int_{-\infty}^{+\infty} dx\, e^{ikx} g(x)$,

with k real, no poles on the real axis.

Providing $g(z)$ satisfies certain conditions, we can evaluate this type of integral as a contour integral in the z-plane, closing the contour with a semicircle $R \to \infty$ in the upper half plane if $k > 0$ and with a semicircle in the lower half plane if $k < 0$.

$$e^{ikz} \longrightarrow e^{ikR\cos\theta}\, e^{-kR\sin\theta}$$

so $|e^{ikz}| \longrightarrow e^{-kR\sin\theta}$ on semicircle.

This is ≤ 1 for semicircle in upper half plane if $k > 0$ and for " " lower " " if $k < 0$. So clearly if $|g(z)| \longrightarrow 0$ at least as fast as $\frac{1}{|z|^2}$, then we just have same case as before for a rational integrand.

But here the condition on g is even less restrictive.

Jordan's lemma: If $g(z)$ converges uniformly to 0 whenever $z \to \infty$, then

$$\lim_{R\to\infty} \int_{C_R} dz\, e^{ikz} g(z) = 0$$

for k a positive real number and C_R the upper half of the circle $|z| = R$.

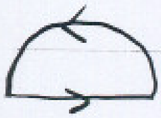
For k a negative real number, we find

$$\lim_{R\to\infty} \int_{C_{R\,lower}} dz\, e^{ikz} g(z) = 0$$

for $g(z) \to 0$ uniformly as $|z| \to \infty$.

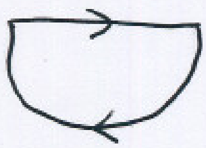
Jordan's lemma allows the use of the residue theorem to evaluate this type of integral with the results:

$$k>0: \quad \int_{-\infty}^{+\infty} dx\, e^{ikx} g(x) = 2\pi i \sum_{\substack{\text{poles } z_n \text{ in} \\ \text{upper half plane}}} e^{ikz_n} \operatorname{Res} g(z_n)$$



CCW

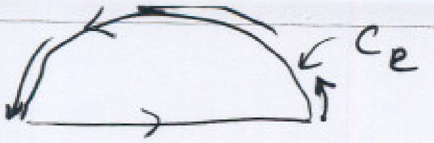
$$k<0: \quad \int_{-\infty}^{+\infty} dx\, e^{ikx} g(x) = -2\pi i \sum_{\substack{\text{poles } z_n \text{ in} \\ \text{lower half plane}}} e^{ikz_n} \operatorname{Res} g(z_n)$$



CW $\to$ sign change

To verify this, we must show that the contribution to the integral along the semicircle vanishes under the condition that g converges to 0 uniformly.

Consider $k>0$. The proof for $k<0$ follows identically.



$z = Re^{i\theta}$
$= R\cos\theta + iR\sin\theta$

The integral along C_R: $\quad dz = iRe^{i\theta}d\theta$

$$I_R = \int_{C_R} dz\, e^{ikz} g(z) = iR\int_0^{\pi} d\theta\, e^{i\theta} e^{ikR\cos\theta} \times$$

$$\times\, e^{-kR\sin\theta} g(Re^{i\theta})$$

Investigate $|I_R| = \left| R\int_0^{\pi} d\theta\, e^{-kR\sin\theta} g(Re^{i\theta})\right|$

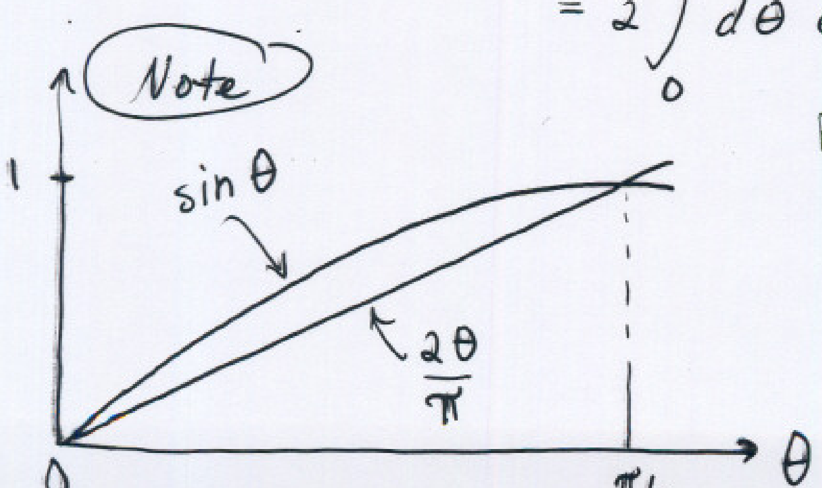
If $g(z)$ converges uniformly as $|z| \to \infty$ (that is, independent of θ) to zero:

$$|g(z)| < \epsilon(R) \qquad \text{and } \epsilon(R) \underset{R\to\infty}{\longrightarrow} 0,$$

then

$$|I_R| \leq R\,\epsilon(R) \underbrace{\int_0^{\pi} d\theta\, e^{-kR\sin\theta}}_{= 2\int_0^{\pi/2} d\theta\, e^{-kR\sin\theta}}$$

use $\int_0^{\pi} = \int_0^{\pi/2} + \int_{\pi/2}^{\pi}$ and $\sin(\pi-\theta) = \sin\theta$



For θ btwn $0 \to \pi/2$, $\sin\theta > \frac{2\theta}{\pi}$

$$\therefore e^{-kR\sin\theta} < e^{-2\theta kR/\pi}$$

Thus $|I_R| \leq 2R\,\epsilon(R) \underbrace{\int_0^{\pi/2} d\theta\, e^{-2kR\theta/\pi}}_{\frac{\pi}{2kR}\left[1-e^{-kR}\right]}$

$$= \frac{\pi}{k}\,\epsilon(R)\left[1-e^{-kR}\right]$$

$$\xrightarrow[R\to\infty]{} 0 \quad \text{if} \quad \epsilon(R) \xrightarrow[R\to\infty]{} 0$$

So, as long as $g(z)$ converges uniformly to 0 as $|z| \to \infty$, Jordan's lemma holds.

Example:

Evaluated at pole at $z = +ib$

$$\int_{-\infty}^{+\infty} dx \, \frac{e^{ikx}}{\underset{\uparrow *}{x^2+b^2}} = 2\pi i \begin{Bmatrix} \dfrac{e^{-kb}}{2ib} & k>0 \\ -\dfrac{e^{kb}}{(-2ib)} & k<0 \end{Bmatrix} = \frac{\pi}{b} e^{-|k|b}$$

Evaluated at pole at $z=-ib$

* Use this to get integrals over $\sin kx\, g(x)$; $\cos kx\, g(x)$

A slight change to the integrand of this example introduces a new type of problem.

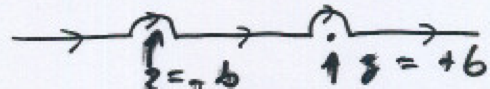
5. Poles on the real axis

Consider $\displaystyle\int_{-\infty}^{+\infty} dx \, \frac{e^{ikx}}{\underset{\uparrow *}{x^2-b^2}}$. In order to evaluate this, we must specify a contour.

This integrand has two simple poles on the real axis — at $x = \pm b$.

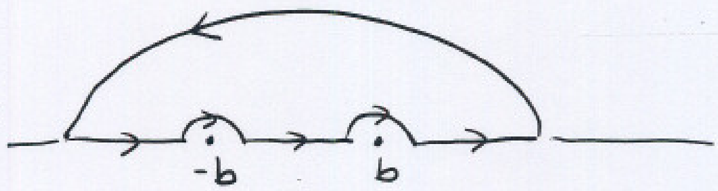
Indent the contour to go around the poles. Can do this in a variety of ways, applying the residue theorem.

(1) $\displaystyle \oint_C dz \, \frac{e^{ikz}}{z^2 - b^2}$ go above the poles



For $k > 0$, must close contour in upper half plane.

$k > 0$: $C =$

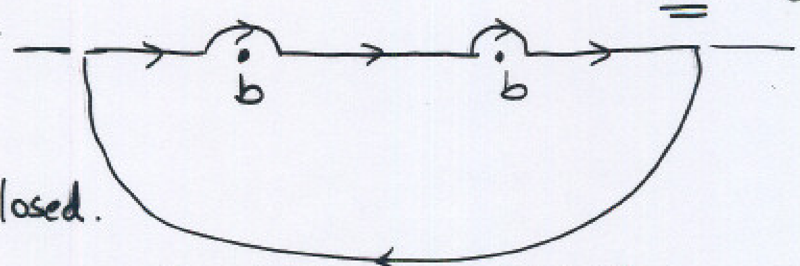


No poles are enclosed by this contour so $\oint = 0$

For $k < 0$, must close contour in lower half plane.

* This is a <u>cw</u> contour

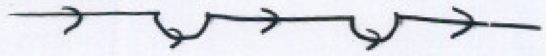
$k < 0$ $C =$



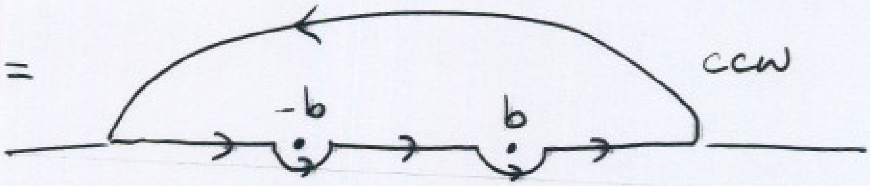
Both poles enclosed.

$$\oint = \underset{\text{cw}}{\underset{\uparrow}{-}} 2\pi i \sum \text{Res} = -2\pi i \left[\frac{e^{ikb}}{(2b)} + \frac{e^{-ikb}}{(-2b)} \right] = \frac{2\pi}{b} \sin kb$$

(2) go below the poles



For $k>0$: $C =$

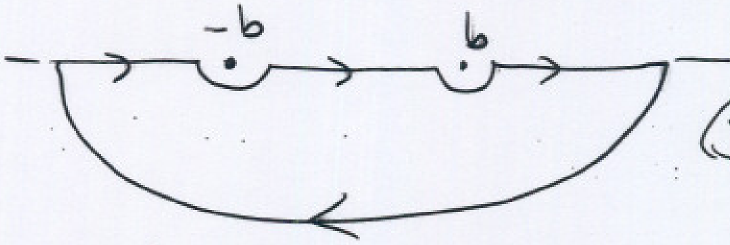


ccw

Both poles enclosed:

$$\oint = 2\pi i \sum \text{Res} = 2\pi i\left[\frac{e^{ikb}}{(2b)} + \frac{e^{-ikb}}{(-2b)}\right] = -\frac{2\pi}{b}\sin kb$$

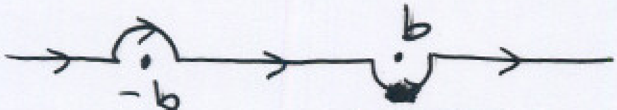
For $k<0$: $C =$



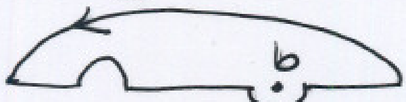
* cw

No poles enclosed. $\oint = 0$

(3)



$k>0$: $C =$



Pole at $x=b$ enclosed.

$$\oint = 2\pi i\, \text{Res}(x=b) = 2\pi i \frac{e^{ikb}}{(2b)} = \frac{i\pi}{b} e^{ikb}$$

$k<0$: $C =$

* cw

Pole at $x=-b$ enclosed.

$$\oint = -2\pi i\, \text{Res}(x=-b) = -2\pi i \frac{e^{-ikb}}{(-2b)} = \frac{i\pi}{b} e^{-ikb}$$

cw

$$\oint_C dz \frac{e^{\cdot}}{(z^2-b^2)}$$

Each of these contour integrals has a different answer!

But we started out looking for:

$$\int_{-\infty}^{+\infty} dx \frac{e^{ikx}}{x^2-b^2}$$

This integral is simply ambiguous. Unless we have a physics context to tell us how to form our contour, the integral is meaningless.

Classic example to illustrate the print:

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x} dx$$ ← Nominally looks like a simple pole on the real axis at $x=0$ but this is one of those removable singularities so this integral is perfectly well defined.

So try continuing to the complex plane and using residue theory.

$$\rightarrow \oint \frac{\sin z}{z} dz = \frac{1}{2i}\left[\oint dz \frac{e^{+iz}}{z} - \oint dz \frac{e^{-iz}}{z} \right]$$

must close in upper half plane

must close in lower half plane

Often (not always) the appropriate [Wyld chooses] to calculate is the principal value integral around $x=0$

C_1

$\oint_{-\infty}^{+\infty} f(x)\,dx$ $f(z)$ is some function that has a simple pole on the real axis $= at\ z = x = 0$

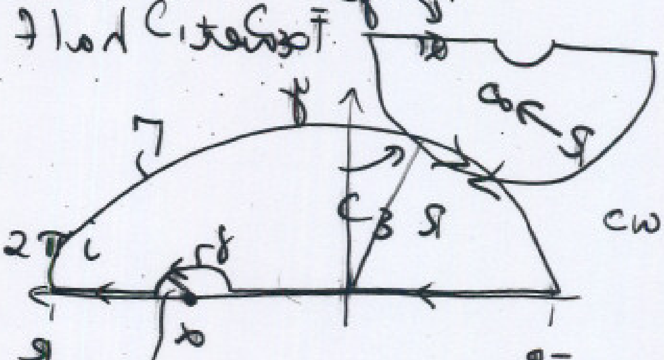
$\oint_{C_1} dz \frac{e^{iz}}{z}$

$C_1 + C_2 =$

C_2

$f(z)$ also falls off sufficiently nicely at large $|z|$ such that the integral of a large semicircle vanishes ($\int dz \frac{e^{iz}}{z}$ in either upper or lower half plane depending on z).

C_1+C_3



The closed contour:

$\oint_{C_1+C_2} dz \frac{e^{iz}}{z} = 2\pi i\, \mathrm{Res}(z=0) = 2\pi i$

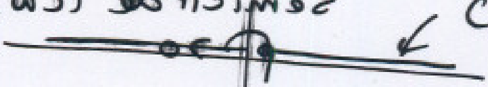
several parts:

$\oint_{C_1+C_3} dz \frac{e^{-iz}}{z} = 2\pi i\,(0) = 0$

$\Rightarrow \int_{-\infty}^{+\infty} \frac{\sin x}{x}\,dx = \frac{1}{2i}\left[\int_{-R}^{-\rho} + \int_{\rho}^{R} + 2\pi i\right] = \pi$

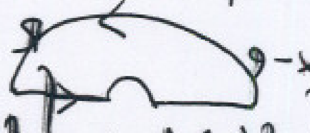
small semicircle (cw)

Try this instead:

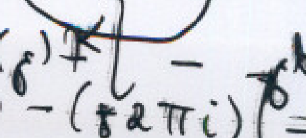


C_1'

$\oint_{C_1'+C_2} dz \frac{e^{iz}}{z} = \left[\int f(z)\,dz + \int f(z)\,dz\right] = \lim \oint_{-\infty}^{+\infty} f(x)\,dx = -2\pi i\, \mathrm{Res}(z=0) = -2\pi i$



$\oint_{C_1'+C_3} dz \frac{e^{-iz}}{z} = \oint dz \frac{e^{-iz}}{z} =$



$\Rightarrow \int \frac{\sin x}{x}\,dx = \frac{1}{2i}\left[0 - (-2\pi i)\right] = \pi$

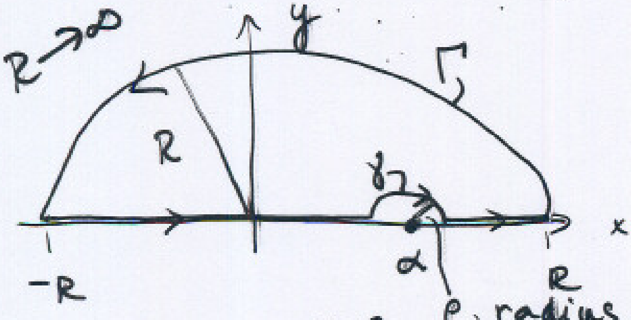
Same Result!

Often (not always) the appropriate integral to calculate is the principal value integral

$$\mathcal{P}\int_{-\infty}^{+\infty} f(x)\,dx$$

$f(z)$ is some function that has a simple pole on the real axis at $z = x = \alpha$.

$f(z)$ also falls off sufficiently nicely at large $|z|$ such that the integral on a large semicircle vanishes (in either upper or lower half plane, depending on f).



The closed contour C is made up of several parts:

$\rho \to 0$

$$\oint_C = \int_{-R}^{\alpha-\rho} + \int_{\gamma} + \int_{\alpha+\rho}^{R} + \cancel{\int_{\Gamma}}$$

γ, small semicircle (cw), $\rho \to 0$

Γ, large semicircle (ccw), $R \to \infty$

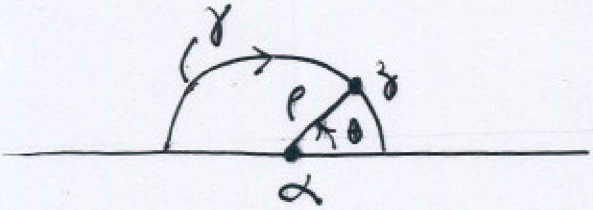
$f(z)$ is such that this vanishes.

$$\mathcal{P}\int_{-\infty}^{+\infty} f(x)\,dx \equiv \lim_{\substack{R\to\infty \\ \rho\to 0}} \left[\int_{-R}^{\alpha-\rho} f(z)\,dz + \int_{\alpha+\rho}^{R} f(z)\,dz\right]$$

$$= \oint_C f(z)\,dz - \int_\gamma f(z)\,dz = 2\pi i \sum_{y>0} \operatorname{Res} f(z) - \int_{\gamma} f(z)\,dz$$

$y>0$: upper half plane; $\rho \to 0$

To find the principal value integral, we must evaluate the integral around the small semicircle in the limit $\rho \to 0$.



On γ, $z = \alpha + \rho e^{i\theta}$

$dz = i\rho e^{i\theta} d\theta$

$\boxed{z - \alpha = \rho e^{i\theta}}$

$$\lim_{\rho \to 0} \int_\gamma f(z)\,dz = \lim_{\rho \to 0} i \int_\pi^0 f(\alpha + \rho e^{i\theta}) \rho e^{i\theta} d\theta$$

Recall $f(z)$ has a simple pole at $z = x = \alpha$. So we can express it as:

Residue of $f(z)$ at α.

$$f(z) = \frac{\varphi(z)}{(z-\alpha)} + \psi(z)$$

where both $\varphi(z)$ and $\psi(z)$ are analytic at $z = \alpha$.

In the limit $\rho \to 0$, $\psi(z)$ will not contribute since

$$\lim_{\rho \to 0} i \int \psi(\alpha + \rho e^{i\theta}) \rho e^{i\theta} d\theta \longrightarrow 0$$

The remaining integral is

$$\lim_{\rho \to 0} i \int_\pi^0 \frac{\phi(z)}{(z-\alpha)} \underbrace{\boxed{(\rho e^{i\theta})}}_{= z-\alpha} d\theta = i \lim_{\rho \to 0} \int_\pi^0 \varphi(\alpha + \rho e^{i\theta}) d\theta$$

$$= i\varphi(\alpha) \int_\pi^0 d\theta = -i\pi\varphi(\alpha) = -i\pi \operatorname{Res} f(\alpha)$$

For contour closed in upper plane: (9)

$$P\int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i \sum_{y>0} \text{Res}\, f(z) + i\pi \sum_{y=0} \text{Res}\, f(z)$$

generalizing ↗ to include all simple poles on the real axis.

Return to $\int_{-\infty}^{+\infty} \frac{\sin x}{x}\,dx$:

$$P\left[\,\text{Im}\int_{-\infty}^{+\infty} dx\,\frac{e^{ix}}{x}\right] = \text{Im}(i\pi) = \pi \quad ✓$$

only singularity is the pole at $x=0$.

or $P\left[\frac{1}{2i}\left(\int_{-\infty}^{+\infty} dx\,\frac{e^{ix}}{x} - \int_{-\infty}^{+\infty} dx\,\frac{e^{-ix}}{x}\right)\right]$ → must close in lower half

$$= \frac{1}{2i}\Big[(0+i\pi) - \big(-2\pi i(1) + i\pi(1)\big)\Big]$$

$= -2\pi i\, \text{Res}_{\text{"}y<0\text{"}} + i\pi\, \text{Res}(y=0)$

$= \pi$ ✓

or Could also choose → ⌣ → - - - - -

So we get complete consistency in evaluating the well defined integral

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x}\,dx .$$

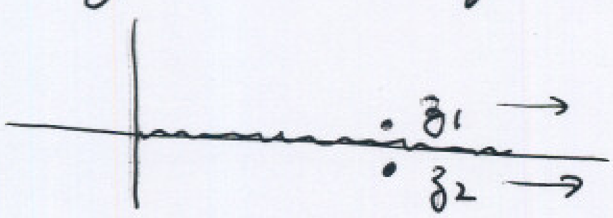
6. Integrals on the range $0 \to \infty$

In order to evaluate the integral of a function that is not even over $0 \to \infty$, we can exploit the multivalued function $\log z$ and its branch cut structure.

We want $\int_0^\infty dx\, f(x)$.

We're going to consider $\oint dz\, f(z) \log z$.
Assume $f(z)$ has simple poles z_n, where n is finite.
Recall $\log z = \log(re^{i\theta}) = \ln r + i\theta$

Restrict to one branch — say $0 \leq \theta < 2\pi$. This puts the branch cut, at which $\log z$ has a discontinuity along the positive real axis, our integration range in the original real integral.

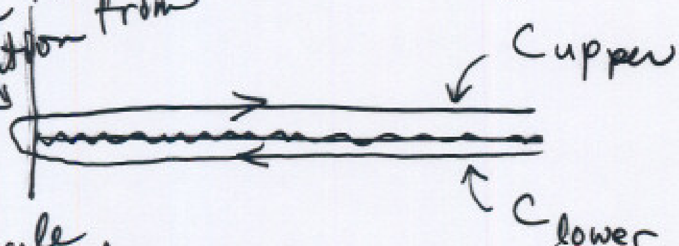


$z_1 \to \theta = 0 \quad \log z_1 = \ln r$

$z_2 \to \theta = 2\pi \quad \log z_2 = \ln r + 2\pi i$

Consider the following contour:

Assume no contribution from this little semicircle around the branch point.



Check out:

$$\int_{C_{lower} + C_{upper}} dz\, f(z) \log z$$

Along C_{lower}, $\theta = 2\pi$, so $\log z = \ln r + 2\pi i$

Along C_{upper}, $\theta \to 0$, so $\log z = \ln r$.

$$\therefore \int\limits_{C_{lower}+C_{upper}} dz\, f(z) \log z = \int_{\infty}^{0} dr\, f(r)\left[\ln r + 2\pi i\right] \quad \leftarrow \text{lower}$$

$$+ \int_{0}^{\infty} dr\, f(r)\left[\ln r\right] \quad \leftarrow \text{upper}$$

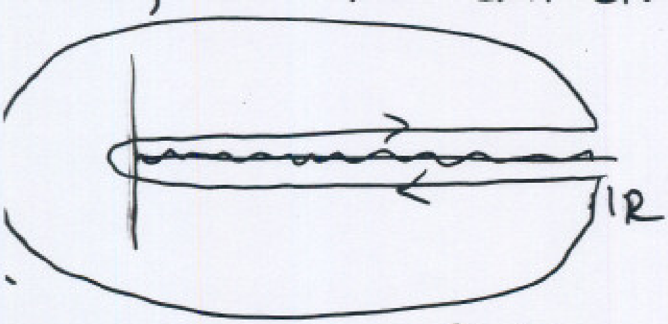
$$= -\int_{0}^{\infty} dr\, f(r)\left[\ln r + 2\pi i\right] + \int_{0}^{\infty} dr\, f(r) \ln r$$

cancel

So now we have:

$$\int\limits_{\substack{C_{lower} + \\ C_{upper}}} dz\, f(z) \log z = -2\pi i \underbrace{\int_{0}^{\infty} dx\, f(x)}_{\text{This is what we want.}}$$

Now, if we can close this contour as:



$R \to \infty$

then as long as $R \ln R |f(R)|$ vanishes as $R \to \infty$, we can use this closed contour and apply the residue theorem.

Thus,

$$\int_0^\infty dx\, f(x) = -\frac{1}{2\pi i} \oint_{\ominus} dz\, f(z) \log z$$

$$= -\sum_n \text{Res } f(z_n) \log z_n$$

Example $\quad f(x) = \dfrac{1}{1+x^m} \qquad m = 2, 3, \ldots$

This is of the form $\dfrac{p(x)}{q(x)}$ so the residue at pole z_i is $\dfrac{p(z_i)}{q'(z_i)}$ where $p = 1$ and $q = 1 + z^m$.

$\text{Res } f(z_i) = \dfrac{1}{m z_i^{m-1}}$

The poles are at the mth roots of -1.

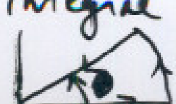
$$z_n = e^{i(\pi + 2\pi n)/m} \qquad n = 0, 1, \ldots$$

For $m = 3$, $\quad z_1 = e^{i\pi/3}$; $z_2 = e^{i\pi}$; $z_3 = e^{i5\pi/3}$

$$\int_0^\infty dx \frac{1}{1+x^{\underset{\uparrow}{3}}} = \frac{-1}{\underset{\uparrow}{3}}\left[\frac{i\pi/3}{e^{\underset{\uparrow}{2}i\pi/3}} + \frac{i\pi}{e^{\underset{\uparrow}{2}i\pi}} + \frac{i5\pi/3}{e^{\underset{\uparrow}{2}(5i\pi/3)}}\right]$$

$\qquad m \qquad m \qquad m-1 \qquad m-1 \qquad m-1$

$$= \frac{2\pi\sqrt{3}}{9}$$

[Note: a different way to do this integral is using pie-slice contour  $\theta = \alpha\pi/m \rightarrow$ chosen to enclose 1 pole.

$R \to \infty$

Result $\displaystyle\int_0^\infty \frac{dx}{1+x^m} = \frac{\pi}{m \sin(\pi/m)}$] $\quad \sin(\pi/3) = \sqrt{3}/2$ ✓

Another example using the branch cut of a multivalued function to define a useful contour is integrals of the form

$$\int_0^\infty x^{\lambda-1} R(x)\, dx$$

λ is not an integer (eg $\sqrt{x}$)

Here $R(z)$ must be rational, analytic at $z=0$, and it cannot have poles on the real axis.

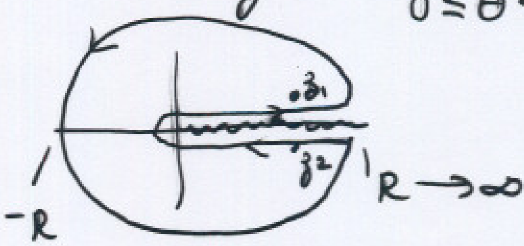
Also, we need $|z^\lambda R(z)| \longrightarrow 0$ uniformly as $|z| \to 0$ and $|z| \to \infty$.

Since λ is noninteger, $z^{\lambda-1}$ is a multivalued function in general. The form of the power, $\lambda-1$, was convenient to state the limit properties but let's change notation now.

$$\int_0^\infty x^\alpha R(x)\, dx$$

α noninteger. $0 \le \theta < 2\pi$

Consider again the contour



At z_1 on the part of the contour above real axis, $\theta = 0$, $z^\alpha = r^\alpha$

At z_2, $\theta = 2\pi$, so $z^\alpha = r^\alpha e^{2\pi i\alpha}$

The contributions along the curved segments vanish due to the limiting behaviour of the integrand.

Thus $\oint z^\alpha R(z)\,dz = \underbrace{\int_\infty^0 z^\alpha R(z)\,dz}_{\text{lower}} + \underbrace{\int_0^\infty z^\alpha R(z)\,dz}_{\text{upper}}$

$$\int_\infty^0 dx\, x^\alpha e^{2\pi i\alpha} R(x) \qquad\qquad \underbrace{\int_0^\infty x^\alpha R(x)\,dx}_{\text{what we want}}$$

$$= -\int_0^\infty dx\ x^\alpha R(x)\,\left(e^{2\pi i\alpha}\right)$$

$$\therefore \int_0^\infty dx\ x^\alpha R(x) = \frac{1}{(1-e^{2\pi i\alpha})}\underbrace{\oint dz\, z^\alpha R(z)}_{= 2\pi i \sum_n (\operatorname{Res} R(z_n)) z_n^\alpha}$$

Analytic Continuation

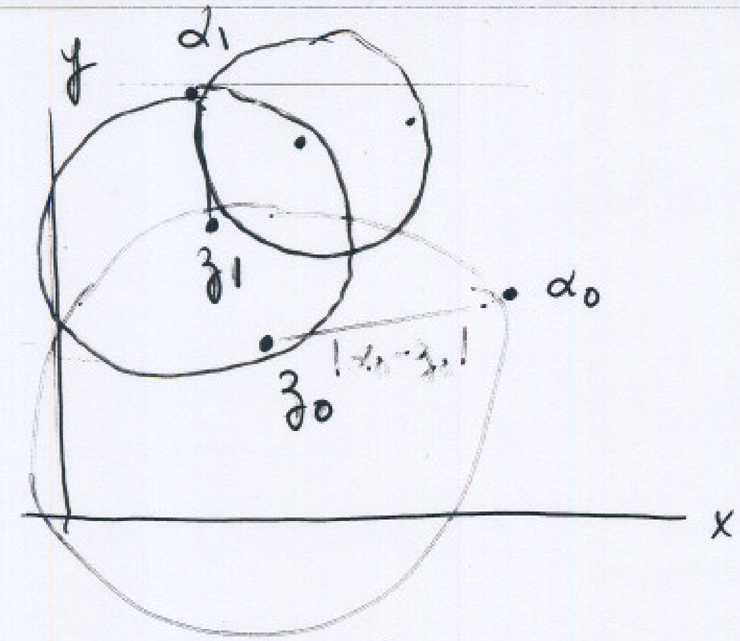
Wyld calls this the "ultimate fabulous property of analytic functions. It demonstrates the rigid structure of analyticity. As we've seen, the property of analyticity places restrictions on a function:

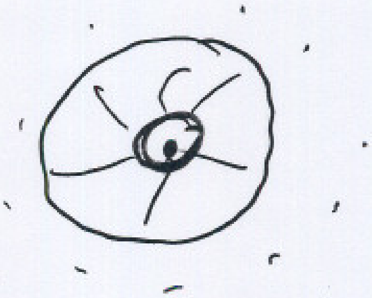
- real and imaginary parts each separately satisfy Laplace's eqn

- real and imaginary parts related via the Cauchy-Riemann eqns.

An analytic function has well defined derivatives to all orders. It can be expressed as a power series expansion. The region in which an expansion is valid is limited by the presence of singularities.

Expand $f(z)$ about z_0. If $f(z)$ is singular at $z = \alpha_0$, the expansion is valid within the region about z_0 of radius $|\alpha_0 - z_0|$. For any point within that region the function and its derivatives exist. So we can make another series expansion about any point in that region → for instance choose z_1 near the border of the region.





The expansion of $f(z)$ about z_1 is valid up to the nearest singularity. This defines an extended region of analyticity. The function is continued into the larger reason.

Continuing from the knowledge of $f(z)$ in the neighbourhood, no matter how small, of z_0, can determine the function throughout the whole complex plane, excluding points where $f(z)$ is singular.

→ analytic continuation ←

The values of a function in any region of the complex plane determine its values everywhere that it is analytic.

Analytic continuation only fails if the function has a continuous line of singularities such that you can't get around it.
→ Continue off the real axis → preserves many identities, properties