Green's Functions – Solving Inhomogeneous Differential Equations $\rightarrow$ Sources

- Poisson $\qquad \nabla^2 \phi = -4\pi\rho$
- Heat / Diffusion $\qquad \nabla^2 u - \frac{1}{K}\frac{\partial u}{\partial t} = -4\pi\rho$
- Wave $\qquad \nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = -4\pi\rho$

These inhomogeneous equations describe situations where there is

- a charge (or mass) density
- a heat source (or matter source)
- a driven oscillation

The method we will explore to solve such equations is that using Green's functions. It relies on the powerful tool of superposition that we can use for linear problems.

Put in our $\mathcal{S}\mathcal{L}$ form, our 1-d problem looks like

$$\mathcal{L}u(x) = \underbrace{\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] - q(x)u(x)}_{\text{usual self-adjoint form}} = \underbrace{\phi(x)}_{\text{some given source}}$$

As always, part of the description of the problem is the statement of the range and the boundary conditions.

Consider interval $a \le x \le b$

b.c.'s at a and b

The Green function is the solution to our equation for the case of a point source located at x'.

$$\mathcal{L}\, G(x, x') = \delta(x - x')$$

(We will explore the b.c.s to be imposed on $G(x, x')$.)

The basic idea is that, if we can solve for $G(x, x')$, then we can use superposition to describe the actual source as

$$\phi(x) = \int dx' \, \delta(x - x') \, \phi(x')$$

yielding the solution for the full problem

as "$u(x) = \int dx' \, G(x, x') \, \phi(x')$"

specific form depends on b.c.s

Work through the details:

We have repeatedly found the following construction useful:

$$\langle v | \mathcal{L} u\rangle - \langle u | \mathcal{L} v\rangle = \int_a^b dx \left[v(\mathcal{L} u) - u(\mathcal{L} v)\right]$$

$$= \left[p(x)\left(v \frac{du}{dx} - u \frac{dv}{dx}\right)\right]\Big|_a^b$$

(pg 78 of transparencies, 2.8.8 Wyld)

(sometimes referred to as the Green's theorem identity)

We can use it again here to express a solution $u(x)$ to our nonhomogeneous problem in terms of the Green fn $G(x,x')$. Then, if we can find $G(x,x')$, the whole problem is solved.

In the G's thm identity, choose $v = G(x,x')$.

$$\int_a^b dx \left[G(x,x') \underbrace{(\mathcal{L} u)}_{=\phi(x)} - u \underbrace{(\mathcal{L} \{G(x,x')\})}_{=\delta(x-x')}\right] =$$

$$= \left[\int_a^b dx\, G(x,x')\phi(x)\right] - u(x') = \left[p(x)\left(G(x,x') \frac{du}{dx} - u \frac{d\,G(x,x')}{dx}\right)\right]\Big|_a^b$$

To write an expression for $u(x)$, just exchange the roles of $x \leftrightarrow x'$.

$$u(x) = \int_a^b dx'\, G(x',x)\,\varphi(x') - \left[p(x') \left(\underbrace{G(x',x) \frac{du(x')}{dx'}}_{①} - u(x') \underbrace{\frac{dG(x',x)}{dx'}}_{②} \right) \right]_{x'=a}^{x'=b}$$

Now consider various b.c.s on our solution $u(x)$ of the type we tend to encounter. Our goal, assuming we can find $G(x,x')$, is to choose b.c.'s for G such that we have the expression above for $u(x)$ given entirely in terms of known quantities.

So φ and G are known.

1) $u(a)$ and $u(b)$ are given.

In this case, term ② in $u(x)$ is known but term ① is not specified. If we choose a particular form for the b.c. on $G(x,x')$, we can arrange to dump term ①.

Choose $G(a,x) = G(b,x) = 0$

$$\text{Then } u(x) = \int_a^b dx'\, G(x',x)\,\varphi(x') + \left[p(x')\, u(x') \frac{dG(x',x)}{dx'} \right] \Bigg|_{x'=a}^{b}$$

G, φ, $u(a)$, $u(b)$ all known so this is the solution.

The procedure is to choose b.c.s for G that eliminate any unspecified terms in the expression for u.

2) If $u(a)$ and $\left.\dfrac{du}{dx}\right|_{x=b}$ are known,

choose $\quad G(a,x)=0 \quad$ and $\quad \left.\dfrac{dG(x',x)}{dx'}\right|_{x'=b}=0.$

Then

$$u(x)=\int_a^b dx'\, G(x',x)\,\varphi(x') - p(b)\,G(b,x)\left.\frac{du(x')}{dx'}\right|_{x'=b}$$

$$- p(a)\,u(a)\left.\frac{dG(x',x)}{dx'}\right|_{x'=a}$$

3) General nonhomogeneous S-L type b.c.:

$$\left.\begin{aligned} A\,u(a)+B\,u'(a) &= X \\ C\,u(b)+D\,u'(b) &= Y \end{aligned}\right\} \text{ these combinations are known}$$

The form of the boundary terms in expression for $u(x)$ is:

$$-p(b)\Big(G(b,x)\overbrace{u'(b)}^{=\frac{1}{D}(Y-C\,u(b))} - u(b)\,G'(b,x)\Big)$$

$$+\,p(a)\Big(G(a,x)\underbrace{u'(a)}_{=\frac{1}{B}(X-A\,u(a))} - u(a)\,G'(a,x)\Big)$$

Focus on the term at $x' = a$:

$$p(a)\Big(\underbrace{G(a,x)\frac{X}{B}}_{\text{all Known}} - \underbrace{\left[\frac{A}{B}G(a,x) + G'(a,x)\right]}_{\text{set this to zero}}\underbrace{u(a)}_{\text{unKnown}}$$

Choose $\quad A\,G(a,x) + B\,G'(a,x) = 0$

Similarly $\quad C\,G(b,x) + D\,G'(b,x) = 0$

$$\left(G'(\alpha,x) \equiv \frac{dG(x',x)}{dx'}\bigg|_{x'=\alpha} \right)$$

With this choice of b.c.s for G, $u(x)$ can be expressed in terms of Known quantities as

$$u(x) = \int_a^b dx'\, G(x',x)\,\varphi(x') \underbrace{- p(b)G(b,x)\frac{Y}{D} + p(a)G(a,x)\frac{X}{B}}$$

$$\underline{\underline{\text{or}}} \quad = p(b)G'(b,x)\frac{Y}{C} - p(a)G'(a,x)\frac{X}{A}$$

We choose b.c.s for $G(x',x)$ to yield a solution $u(x)$ that is completely determined.

Before developing method to find $G(x,x')$, we will note a symmetry property:

$$G(x,x') = G(x',x)$$

Show this by using the Green's thm identity on pg 237 with $u = G(x,x')$ and $v = G(x,x'')$.

Assume we impose general S-L boundary conditions at $x=a$ and b, as in the last case considered

$$\int_a^b dx \; G(x,x'')\Big(\underbrace{\mathcal{L}\,G(x,x')}_{=\delta(x-x')}\Big) - \int_a^b dx \; G(x,x')\Big(\underbrace{\mathcal{L}\,G(x,x'')}_{=\delta(x-x'')}\Big)$$

$$= G(x',x'') - G(x'',x')$$

$$= \left[p(x)\left(\left(G(x,x'')\frac{d}{dx}G(x,x') - G(x,x')\frac{d}{dx}G(x,x'')\right)\right)\right]\Bigg|_{x=a}^{x=b}$$

Evaluating ( ) at $x=a$ and $x=b$, similar forms.

$x=a$: $\quad G(a,x'')\left(-\frac{A}{B}G(a,x')\right) - G(a,x')\left(-\frac{A}{B}G(a,x'')\right) = 0$

Thus $\quad G(x',x'') = G(x'',x')$

symmetric under interchange

Will investigate two methods to find Green's fn. One is often known as "division of regions" wherein the regions that do not include the point source (i.e. $x = x'$) are investigated and then the solutions are matched via a particular method. The other way to determine Green's functions relies on expansion as a superposition of eigenfunctions.

We wish to find $G(x,x')$ that satisfies

$$\mathcal{L}\, G(x,x') = \frac{d}{dx}\left[p(x)\,\frac{dG(x,x')}{dx}\right] - q(x)\,G(x,x') = \delta(x-x')$$

For $x \neq x'$, this simply reduces to a homogeneous SL problem that we can solve. That is, solve homo problem in interval $x < x'$, then in the interval $x > x'$, and match the solutions.

For the matching conditions focus on an infinitesmal range in x about x' (which is considered fixed).

$$x' - \epsilon \leq x \leq x' + \epsilon \quad \text{with} \quad \epsilon \longrightarrow 0$$

Integrate the differential equation for $G(x,x')$

in x over this range.

$$\int_{x'-\epsilon}^{x'+\epsilon} dx \left\{ \frac{d}{dx}\left[p(x)\,\frac{dG(x,x')}{dx}\right] - q(x)\,G(x,x') \right\} = \int_{x'-\epsilon}^{x'+\epsilon} dx\ \delta(x-x') = 1$$

$$\parallel$$

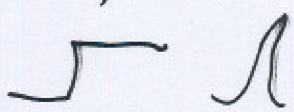
$$\left[\underbrace{p(x)}_{p(x'+\epsilon)}\,\frac{dG(x,x')}{dx}\right]\Bigg|_{x=x'+\epsilon} - \left[\underbrace{p(x)}_{p(x'-\epsilon)}\,\frac{dG(x,x')}{dx}\right]\Bigg|_{x=x'-\epsilon} - \underbrace{\int_{x'-\epsilon}^{x'+\epsilon} dx\ q(x)\,G(x,x')}$$

Assume $p(x)$ is continuous at $x=x'$. Then for $\epsilon \to 0$, take out this common factor as $p(x')$.

if the integrand is finite in this region, this integral will be $\leq \max\left|q(x)G(x,x')\right|(2\epsilon) \longrightarrow 0$ as $\epsilon \to 0$.

$$\therefore \quad \boxed{\frac{dG(x,x')}{dx}\Bigg|_{x=x'-\epsilon}^{x=x'+\epsilon} = \frac{1}{p(x')}}$$

This is one condition on $G(x,x')$.



Also we insist $G(x,x')$ is continuous at $x=x'$. Otherwise, we would find a contradiction in the d.e. that it must satisfy. That is, if G were discontinuous at $x=x'$, then its first derivative must be $\frac{dG}{dx} \sim \delta(x-x')$. In this case the $\frac{d^2G}{dx^2}$ term should generate a term in $\frac{d}{dx}\delta(x-x')$. But there is no such term in the 2nd order d.e. that G satisfies.

$$\therefore \quad G(x'+\epsilon, x') = G(x'-\epsilon, x') \quad \text{as } \epsilon \to 0.$$

This is a second condition on $G(x,x')$. These last two conditions constitute continuity conditions.

Procedure to find $G(x,x')$:

- Solve $\dfrac{d}{dx}\left[p(x)\dfrac{d}{dx}G(x,x')\right] - q(x)\,G(x,x') = 0$

  for $x < x'$ and for $x > x'$. This is a 2nd order d.e. so there will be two integration constants for each region.

- Apply the boundary conditions (as implied by the corresponding conditions on $u(x)$)

$$\boxed{\begin{aligned} &A\,G(a,x) + B\frac{d}{dx'}G(x',x)\Big|_{x'=a} = 0 \\ &C\,G(b,x) + D\frac{d}{dx'}G(x',x)\Big|_{x'=b} = 0 \end{aligned}}$$

- Apply the continuity conditions:

$$\boxed{\begin{aligned} &\frac{dG}{dx}(x,x')\Bigg|_{x=x'-\epsilon}^{x=x'+\epsilon} = \frac{1}{p(x')} \\ &G(x'+\epsilon, x') = G(x'-\epsilon, x') \quad \epsilon \to 0 \end{aligned}}$$

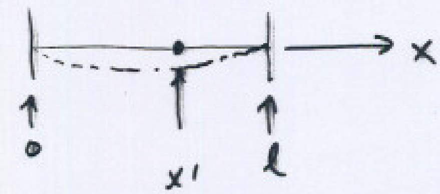
The four conditions allow the four constants to be determined.

The result is a piecewise continuous solution.

Specific example to illustrate the method:

- based on 1-d wave equation

• bowed stretched string



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transverse force acting on string.

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} = f(x,t)$$

Assume $f(x,t) = f(x)\, e^{-i\omega t}$. The general case is considerably more complex. In this simpler case, $u(x,t)$ with the same $e^{-i\omega t}$ time dependence will satisfy the equation.

$$u(x,t) = u(x)\, e^{-i\omega t}$$

Response same as driving force.

$$\frac{d^2 u(x)}{dx^2} + \underbrace{\boxed{\frac{\omega^2}{c^2}}}_{\equiv k^2} u(x) = f(x)$$

→ nonhomogeneous Helmholtz equation

This is S-L form with $p(x) = 1$.

In general, can use the normal mode solutions to the corresponding homogeneous problem to satisfy any initial conditions that are imposed.

Focus on the non homogeneous problem.

Assume fixed end boundary conditions:

$$u(0) = u(l) = 0$$

This is an example of our Case 1 so the corresponding boundary conditions on the Green's function are:

$$G(0, x') = G(l, x') = 0$$

$G(x, x')$ satisfies the equation

$$\frac{d^2 G(x,x')}{dx^2} + k^2 G(x,x') = \delta(x - x')$$

$= 0$ for $x \neq x'$

$x < x'$: $\quad G(x,x') = A \sin kx + B \cos kx$

$x > x'$: $\quad G(x,x') = C \sin kx + D \cos kx$

$G(0,x') = 0 \Longrightarrow B = 0 \quad \therefore G(x,x') = A \sin kx \qquad x < x'$

$G(l,x') = 0 \Longrightarrow C \sin kl + D \cos kl = 0$

$D = -C \tan kl$

$G(x,x') = C[\sin kx - \tan kl \cos kx]$ $\quad \therefore G(x,x') = E \sin k(x-l) \qquad x > x'$

use trig identity

Apply the continuity condition:

$$\frac{dG(x,x')}{dx}\Bigg|_{x=x'-\epsilon}^{x=x'+\epsilon} = \frac{1}{p(x')} = 1$$

$$kE\cos k(x'+\underset{0}{\cancel{\epsilon}}-l) - kA\cos k(x'-\underset{0}{\cancel{\epsilon}}) = 1$$

and the other continuity condition

$$G(x'+\epsilon, x') = G(x'-\epsilon, x')$$

$$E\sin k(x'+\underset{0}{\cancel{\epsilon}}-l) = A\sin k(x'-\underset{0}{\cancel{\epsilon}})$$

- Solve these for A and E

$$A = \frac{\sin k(x'-l)}{k\sin kl} \qquad E = \frac{\sin kx'}{k\sin kl}$$

$$\left.\begin{aligned} x<x'&: \quad G(x,x') = \frac{1}{k\sin kl}\sin kx\,\sin k(x'-l) \\ x'<x&: \quad G(x,x') = \frac{1}{k\sin kl}\sin k(x-l)\,\sin kx' \end{aligned}\right\} = \frac{1}{k\sin kl}\sin kx_<\,\sin k(x_>-l)$$

Now we have the Green's function for this problem so we can find the displacement $u(x)$.

$$u(x) = \int_0^l dx' \, G(x,x') \underbrace{\varphi(x')}_{f(x')} + \left[p(x') \underbrace{u(x')}_{\substack{=0 \\ \text{at } x'=0,l}} \frac{dG(x',x)}{dx'} \right] \Bigg|_{x'=0}^{l}$$

Break $\int_0^l dx'$ into $\underbrace{\int_0^x dx'}_{x'<x} + \underbrace{\int_x^l dx'}_{x'>x}$

$$= \int_0^x dx' \, \frac{\sin k(x-l)}{k \sin kl} \sin kx' \, f(x') + \int_x^l dx' \, \frac{\sin kx}{k \sin kl} \sin k(x'-l) f(x')$$

Now it's "simply" a question of doing these integrals for a particular force.

This result followed from the 1-d wave equation with a bowing force imposed along the length of the string.

Alternately we can adapt the results to a different physical situation where $\boxed{f(x)=0}$ but one end (say $x=l$) of the string is oscillating with frequency ω and amplitude A

$$u(x=l,t) = A e^{-i\omega t}$$

$$u(x=l) = A$$

Here we have the same type of boundary conditions on $u(x)$. $u(x=0)$ is still fixed such that $u(x=0)=0$. Also $u(l)$ is still given but now $u(x=l)=A$. i.e. Case 1

So the corresponding boundary conditions on the Green's fn are the same. Thus the Green's fn we just derived is also appropriate for this case.

Because $f(x) = 0$, only the boundary terms contribute to $u(x)$:

$$u(x) = \left[\underbrace{p(x')}_{=1}\, u(x')\, \frac{dG(x',x)}{dx'} \right] \Bigg|_{x'=0}^{x'=l}$$

Consider only $x'=l$ limit.

At $x'=0$, $u(0) = 0$.

$$= \underbrace{u(l)}_{=A}\, \frac{dG(x',x)}{dx'} \Bigg|_{x'=l}$$

Since we need the $x'=l$ limit, use the $x < x'$ part of the Green's fn solution:

$$\frac{dG(x',x)}{dx'} = \frac{1}{\sin kl} \left[\sin kx \cos k(x'-l) \right]$$

Evaluating the derivative at $x'=l$:

$$\frac{dG(x',x)}{dx'} \Bigg|_{x'=l} = \frac{\sin kx}{\sin kl}$$

$$\therefore \quad u(x,t) = u(x)\, e^{-i\omega t} = A\, \frac{\sin kx}{\sin kl}\, e^{-i\omega t}$$

Our second way to determine Green's fns is to consider them as an <u>expansion in eigenfunctions</u> as follows:

In discussing Green's fns, we have worked with the nonhomogeneous equation

$$\mathcal{L}\, u(x) = \phi(x)$$

But when we developed the homogeneous S-L problem previously we stated it in eigenvalue form

$$\mathcal{L}\, u(x) + \lambda\, w(x)\, u(x) = 0$$

So let's now assume this form for the development of the non-homogeneous problem.

Take $\mathcal{L} \longrightarrow \mathcal{L} + \lambda w(x)$ in what we've done with Green's fns so far.

Thus $G(x, x')$ should satisfy

$$* \quad \mathcal{L}\, G(x, x') + \lambda\, w(x)\, G(x, x') = \delta(x - x')$$

The same formal expressions for $u(x)$ in terms of G will follow as before.

The solutions to the corresponding homogeneous S-L problem will be denoted in the usual eigenfn form as $u_n(x)$ such that

$$\mathcal{L}\, u_n(x) = -\lambda_n\, w\, u_n(x)$$

The $u_n(x)$ form a complete orthogonal set for which we will choose the normalization as

$$\int_a^b dx\ w(x)\ u_n(x)\ u_m(x) = \delta_{mn} \quad \text{for simplicity.}$$

They satisfy $\sum_n w(x)\, u_n(x)\, u_n(x') = \delta(x-x')$ (closure

and obey homogeneous boundary conditions

$$A u_n(a) + B u_n'(a) = 0$$

$$C u_n(b) + D u_n'(b) = 0$$

Assume that $G(x,x')$ can be expressed as an expansion in eigenfunctions of the homogeneous S-L equation. For this purpose, consider x' as fixed and x as variable.

$$G(x,x') = \sum_{n=0}^{\infty} \gamma_n(x')\, u_n(x)$$

Put this assumed form into * on page 250 in order to determine the coefficients $\gamma_n(x')$.

$$* \sum_{n=0}^{\infty} \gamma_n(x')\, \underbrace{\mathcal{L}\, u_n(x)}_{= -\lambda_n w(x)\, u_n(x)} + \lambda w(x) \sum_{n=0}^{\infty} \gamma_n(x')\, u_n(x) = \delta(x-x')$$

$$\sum_{n=0}^{\infty} \gamma_n(x')\, [\lambda - \lambda_n]\, w(x)\, u_n(x) = \delta(x-x')$$

Use the orthonormality of the $u_n(x)$.

Multiply by $u_m(x)$ and integrate over x.

$$\gamma_m(x')(\lambda-\lambda_m) = \int_a^b dx\, \delta(x-x')\, u_m(x) = u_m(x')$$

$$\gamma_m(x') = \frac{u_m(x')}{\lambda-\lambda_m}$$

Thus we can express the Green's fn as an expansion in eigenfunctions of the corresponding homogeneous S-L problem.

$$G(x,x') = \sum_{n=0}^{\infty} \frac{u_n(x)\, u_n(x')}{\lambda-\lambda_n}$$

This expression does satisfy the eq. * for G. We must also check that it satisfies the appropriate boundary conditions imposed on G.

Consider the general type (Case 3) conditions that can be imposed on the solution to the nonhomogeneous problem $u(x)$.

$$A\,u(a) + B\,u'(a) = X$$

$$C\,u(b) + D\,u'(b) = Y$$

We need the corresponding conditions on G, as previously determined:

$$A\, G(a,x') + B \frac{d}{dx} G(x,x') \Big|_{x=a} = 0$$

$$C\, G(b,x') + D \frac{d}{dx} G(x,x') \Big|_{x=b} = 0$$

Inserting our expansion for G:

$$A \sum_{n=0}^{\infty} \frac{u_n(a)\, u_n(x')}{\lambda - \lambda_n} + B \sum_{n=0}^{\infty} \frac{u_n'(a)\, u_n(x')}{\lambda - \lambda_n} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{(\lambda - \lambda_n)} \underbrace{\left[A u_n(a) + B u_n'(a) \right]}_{=0} u_n(x')$$

The u_n are solutions to the homogeneous problem. They obey $A u_n(a) + B u_n'(a) = 0$.

Similarly, at $x=b$, the required b.c. on G is satisfied.

For the simple example of the stretched string with fixed ends at $x=0$ and $x=l$, we can compare the new form of the solution (as an infinite series) with the closed form we obtained by the division of regions method.

First we need the eigenfn solutions to the equivalent homogeneous problem – 1-d wave equation with no external force.

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} \longleftarrow h: \text{homogeneo}$$

$$u_h(x,t) = \sum_n u_n(x)\left[A_n \cos k_n ct + B_n \sin\right.$$

satisfies homogeneous Helmholtz

$$\frac{d^2 u_n(x)}{dx^2} + k_n^2 u_n(x) = 0$$

$$u_n(x) \sim \cos k_n x$$

$$\sin k_n x \longleftarrow u_n(0) = 0 \Rightarrow$$

$$u_n(l) = 0 \Rightarrow$$

$$\therefore u_h(x,t) = \sum_n \underbrace{\sin\left(\frac{n\pi x}{l}\right)}\left[A_n \cos\left(\frac{n\pi ct}{l}\right)\right.$$

$$u_n(x) \sim \alpha \sin\left(\frac{n\pi x}{l}\right)$$

Normalizing the $u_n(x)$ such that $\int_0^l dx$

$$\Rightarrow u_n(x) = \sqrt{\frac{2}{l}}\sin\left(\frac{n\pi x}{l}\right)$$

us

$k_n ct]$

efn

only sineform

$k_n l = n\pi$

$+ B_n \sin\left(\frac{n\pi ct}{l}\right)]$

$\alpha^2 \sin^2\left(\frac{n\pi x}{l}\right) = 1$

Thus
$$G(x,x') = \underbrace{\sum_{n=0}^{\infty}}_{\text{infinite series}} \frac{\left(\frac{2}{l}\right) \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi x'}{l}\right)}{\left[\underbrace{k^2}_{=\lambda} - \left(\frac{n\pi}{l}\right)^2\right]}$$
← expansion in eigenfns

infinite series

$$= \underbrace{\frac{1}{k \sin kl} \sin kx_< \sin k(x_> - l)}_{\text{closed form}}$$
← division of regions

closed form

Recognizing that the two forms of the solution must actually be equal gives us this expansion theorem.

Extend to n-dimensions:

Will use 3-d Poisson's eqn to illustrate the extension. Helmholtz eqn can be treated in the same way. We will also consider 4-d Green's functions for the wave eqn and diffusion eqn, which include time dependence.

Reminder of some math: "Green's theorem"

We have the divergence thm (Gauss's thm) as:

$$\int_V (\vec{\nabla}\cdot\vec{A})\, dV = \oint_S \vec{A}\cdot d\vec{a}$$

Use the identities; with u and v scalar fns,

$$① \quad \vec{\nabla}\cdot(u\vec{\nabla}v) = (\vec{\nabla}u)\cdot(\vec{\nabla}v) + u\nabla^2 v$$

$$② \quad \vec{\nabla}\cdot(v\vec{\nabla}u) = (\vec{\nabla}v)\cdot(\vec{\nabla}u) + v\nabla^2 u$$

in the following:

$$\int_V [① - ②]\,dV = \int_V \left[\vec{\nabla}\cdot(u\vec{\nabla}v) - \vec{\nabla}\cdot(v\vec{\nabla}u)\right]dV$$

$$= \int_V \left[u\nabla^2 v - v\nabla^2 u\right]dV$$

by divergence theorem: $= \oint_S \left[u\vec{\nabla}v - v\vec{\nabla}u\right]\cdot d\vec{a}$

$$= \oint_S \left[u\vec{\nabla}v - v\vec{\nabla}u\right]\cdot\hat{n}\,ds$$

$$\boxed{\int_V \left[u\nabla^2 v - v\nabla^2 u\right]dV = \oint_S \left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right]ds}$$

"Green's theorem"

(Assumes that the derivatives of u and v exist within V and on its bounding surface S)

Consider Poisson's equation

$$\nabla^2 \psi(\vec{r}) = -4\pi \rho(\vec{r})$$

applied to a volume V bounded by S. A boundary condition is given such that either $\psi(\vec{r})$ or $\vec{\nabla}\psi(\vec{r})\cdot\hat{n}$ is specified for $\vec{r}$ on S.

Assume a 3-d Green's function satisfies

$$\nabla^2 G(\vec{r},\vec{r}\,') = \delta^3(\vec{r}-\vec{r}\,')$$

with $\vec{r}, \vec{r}\,'$ within V. The b.c.s on G will depend on those on ψ.

Apply Green's thm in various ways. First, use $u = \psi(\vec{r})$ and $v = G(\vec{r},\vec{r}\,')$.

$$\int_V \Big[\psi(\vec{r})\underbrace{\left(\nabla^2 G(\vec{r},\vec{r}\,')\right)}_{=\,\delta^3(\vec{r}-\vec{r}\,')} - G(\vec{r},\vec{r}\,')\underbrace{\left(\nabla^2\psi(\vec{r})\right)}_{=\,-4\pi\rho(\vec{r})}\Big]\, d^3r =$$

$$= \underbrace{\int_S dA\Big[\psi(\vec{r})\left(\vec{\nabla}G(\vec{r},\vec{r}\,')\cdot\hat{n}\right) - G(\vec{r},\vec{r}\,')\left(\vec{\nabla}\psi(\vec{r})\cdot\hat{n}\right)\Big]}$$

$$\psi(\vec{r}\,') + 4\pi\int d^3r\, G(\vec{r},\vec{r}\,')\rho(\vec{r}) = \int$$

Switch $\vec{r} \longleftrightarrow \vec{r}\,'$ so we can write an expression for $\psi(\vec{r})$ in terms of $G(\vec{r}\,',\vec{r})$.

$$\psi(\vec r) = -4\pi \int_{V'} d^3r'\, G(\vec r\,',\vec r)\,\rho(\vec r\,') + \int_{S'} dA' \Big[\underbrace{\psi(\vec r\,')\, \vec\nabla' G(\vec r\,',\vec r)\cdot\hat n'}_{①} - \underbrace{G(\vec r\,',\vec r)\,\vec\nabla'\psi(\vec r\,')\cdot \hat n'}_{②}\Big]$$

denotes diff. w.r.t. prime variables.

This is the 3-d Poisson's eq. analogue of the expression on pg 238:

$$u(x) = \int_a^b dx'\, G(x',x)\,\varphi(x') - \left[p(x')\left(G(x',x)\frac{du(x')}{dx'} - u(x')\frac{dG(x',x)}{dx'} \right)\right]\Bigg|_{x'=a}^{x'=b}$$

Recall, that, for various given b.c.s on $u(x)$, $\frac{du(x)}{dx}$, we __chose__ b.c.s on $G(x,x')$ in order to simplify the boundary term. Do the same in our 3-d case where the "boundary term" is the surface integral.

Two cases:

1) $\psi(\vec r\,')$ given on $S' \longrightarrow$ Dirichlet conditions

If we choose $G_D(\vec r\,',\vec r) = 0$ for $\vec r\,'$ on S', term ② vanishes, eliminating the unknown dependence on $\vec\nabla'\psi(\vec r\,')\cdot\hat n$. In terms on known quantities,

$$\psi(\vec r) = -4\pi \int_{V'} d^3r'\, G(\vec r\,',\vec r)\,\rho(\vec r\,') + \int_{S'} dA'\, \psi(\vec r\,')\,\frac{\partial G(\vec r\,',\vec r)}{\partial n'}$$

2) $\vec{\nabla}'\psi(\vec{r}')\cdot\hat{n}'\Big|_{S'}$ given $\longrightarrow$ Neumann conditions

Choose $\vec{\nabla}' G_N(\vec{r}',\vec{r})\cdot\hat{n}' = K$, a constant.

Generally we cannot choose $K=0$. This is because by the divergence thm, $\int_V \vec{\nabla}\cdot\vec{A}\,dV = \int_S \vec{A}\cdot d\vec{a}$ with $\vec{A} = \vec{\nabla}' G(\vec{r}',\vec{r})$, we have

surface area

$$\int_{V'} \underbrace{\nabla'^2 G(\vec{r}',\vec{r})}_{\delta^3(\vec{r}'-\vec{r})} dV' = 1 = \int_{S'} \underbrace{\vec{\nabla}' G(\vec{r}',\vec{r})\cdot\hat{n}'}_{K} dA' = K S'$$

So put $K = \frac{1}{S'}$, with S' the area of the bounding surface. In terms of known quantities,

$$\psi(\vec{r}) = -4\pi\int_{V'} d^3r'\, G(\vec{r}',\vec{r})\,\rho(\vec{r}') + \underbrace{\frac{1}{S'}\int dA'\,\psi(\vec{r}')}_{=\langle\psi\rangle_{S'}}$$

$$-\int_{S'} dA'\, G(\vec{r}',\vec{r})\,\vec{\nabla}'\psi(\vec{r}')\cdot\hat{n}'$$

$= \langle\psi\rangle_{S'}$, the average value of ψ on surface. Drop this term since it is just a constant.

So, as in the 1-d case, given $G(\vec{r}',\vec{r})$ and boundary conditions on $\psi(\vec{r})$, we can express $\psi(\vec{r})$ in terms of known quantities. Before exploring methods to find $G(\vec{r},\vec{r}')$ investigate its symmetry property under exchange $\vec{r}\leftrightarrow\vec{r}'$.

Same process as $+d$, use Green's thm choosing $u = G(\tilde{\vec{r}},\vec{r})$ and $v = G(\tilde{\vec{r}},\vec{r}\,')$:

$$\int_V d^3\tilde{r}\left[G(\tilde{\vec{r}},\vec{r})\underbrace{\tilde{\nabla}^2 G(\tilde{\vec{r}},\vec{r}\,')}_{\delta^3(\tilde{\vec{r}}-\vec{r}\,')} - G(\tilde{\vec{r}},\vec{r}\,')\underbrace{\tilde{\nabla}^2 G(\tilde{\vec{r}},\vec{r})}_{\delta^3(\tilde{\vec{r}}-\vec{r})}\right] =$$

$$= G(\vec{r}\,',\vec{r}) - G(\vec{r},\vec{r}\,')$$

$$= \int_{\tilde{S}} d\tilde{A}\left[G(\tilde{\vec{r}},\vec{r})\,\tilde{\vec{\nabla}}G(\tilde{\vec{r}},\vec{r}\,')\cdot\hat{\tilde{n}} - G(\tilde{\vec{r}},\vec{r}\,')\,\tilde{\vec{\nabla}}G(\tilde{\vec{r}},\vec{r})\cdot\hat{\tilde{n}}\right]$$

For Dirichlet conditions, the surface integral vanishes because $G(\tilde{\vec{r}},\vec{r}) = G(\tilde{\vec{r}},\vec{r}\,') = 0$ for $\tilde{\vec{r}}$ on $\tilde{S}$

$$\therefore \quad \boxed{G_D(\vec{r}\,',\vec{r}) = G_D(\vec{r},\vec{r}\,')}$$

For Neumann conditions, the surface integral does not nominally vanish. If the bounding surface is infinitely large, $\tilde{\vec{\nabla}}G\cdot\hat{\tilde{n}} \longrightarrow \frac{1}{\tilde{S}} \longrightarrow 0$ so, in this case $G_N(\vec{r}\,',\vec{r}) = G_N(\vec{r},\vec{r}\,')$. For finite surfaces, one often just imposes this exchange symmetry as an additional condition on G_N.

Consider the electrostatic potential $\varphi(\vec{r})$ due to a point charge q located at $\vec{r}'$. It satisfies Poisson's eqn:

$$\nabla^2 \varphi(\vec{r}) = -4\pi q \; \delta^3(\vec{r}-\vec{r}')$$

But this is just the equation the 3-d Green's function for Laplace's eq. satisfies. So $G(\vec{r},\vec{r}')$ is just proportional to the electrostatic potential at $\vec{r}$ due to a point charge at $\vec{r}'$ — the Coulomb potential.

$$\varphi(\vec{r}) = \frac{q}{|\vec{r}-\vec{r}'|} \qquad \nabla^2 G(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}')$$

$$\therefore \quad \boxed{G(\vec{r},\vec{r}') = \frac{-1}{4\pi} \frac{1}{|\vec{r}-\vec{r}'|} + F(\vec{r},\vec{r}')}$$

where F satisfies $\nabla^2 F(\vec{r},\vec{r}') = 0$.

Any term satisfying Laplace's eq can be added — do so in order to satisfy boundary conditions as necessary.

Given $G(\vec{r},\vec{r}')$, the solutions $\psi(\vec{r})$ to Poisson's equation follow as indicated above. For the case that the bounding surface is at infinity, we can ignore $F(\vec{r},\vec{r}')$ and take

$$G(\vec{r},\vec{r}') = \frac{-1}{4\pi} \frac{1}{|\vec{r}-\vec{r}'|}.$$

soln to
$$\nabla^2 \psi(\vec{r}) = -4\pi \rho(\vec{r})$$

Putting G into our general result for $\psi(\vec{r})$ on pg 258:

$$\psi(\vec{r}) = \int_{V'} d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}\,' - \vec{r}|} + \underbrace{\text{surface integral}}_{\equiv \varphi(\vec{r})}$$

Applying ∇^2 to $\psi(\vec{r})$, and using

$$\nabla^2 \left(\frac{1}{|\vec{r}\,' - \vec{r}|} \right) = -4\pi \delta^3(\vec{r}\,' - \vec{r}),$$ we find

$$\nabla^2 \psi(\vec{r}) = -4\pi \rho(\vec{r}) + \nabla^2 \varphi(\vec{r})$$

This implies that $\nabla^2 \varphi(\vec{r}) = 0$. So we interpret the surface integral contribution as due to infinitely remote sources such as, for instance, would produce a uniform external field. Thus take $\varphi(\vec{r}) = 0$ in the absence of external fields.

So it seems that we have solved the problem of Poisson's eqn in general since we have an expression for the appropriate Green's fn and can then write corresponding solns $\psi(\vec{r})$.

In practice, as is usual with boundary value problems, this is not always easy. A closed form for $G(\vec{r}, \vec{r}\,')$ satisfying all b.c.s may be hard to find + the integrals involved in calculating ψ may be tough more properly: "closure"

First look at a method wherein we utilize our knowledge of completeness relations for complete sets of orthogonal functions to deal with part of the problem and, in so doing, reduce to a 1-d Green's fn problem.

- Poisson's equation – use <u>spherical geometry</u> as an example

$$\nabla^2 G(\vec{r}, \vec{r}\,') = \delta(\vec{r} - \vec{r}\,')$$

Write the δ fn in spherical coordinates:

$$\delta(\vec{r} - \vec{r}\,') = \delta(r - r')\, \frac{\delta(\theta - \theta')}{r'}\, \frac{\delta(\varphi - \varphi')}{r' \sin\theta'}$$

$$= \frac{1}{r'^2}\, \delta(r - r') \left[\frac{\delta(\theta - \theta')\, \delta(\varphi - \varphi')}{\sin\theta'} \right]$$

We will effectively reduce the problem of finding $G(\vec{r}, \vec{r}\,')$ to finding a 1-d Green's fn in the radial coordinate by recognizing that we can express the δ fns in the angles in terms of closure relations for the complete set of orthogonal fns of these angles – the spherical harmonics.

Recall the general form we've had for closure relations in a number of cases:

$$\frac{\sum_{n=0}^{\infty} y_n(x)\, y_n^*(x')\, w(x')}{I_n} = \delta(x-x')$$

with the $y_n(x)$ orthogonal fns and

$$I_n = \int_a^b |(y_n(x))|^2\, w(x)\, dx$$ is their normalization condition.

On a surface of constant radius, we have (see page 178)

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{+l} Y_{lm}(\theta,\varphi)\, Y_{lm}^*(\theta',\varphi') = \frac{\delta(\theta-\theta')\,\delta(\varphi-\varphi')}{\sin\theta'}$$

($I_{lm} = 1$; note complex conjugate for case of complex fns)

Use the above form for the angular δ fns in the eqn for G. Also, express the angular part of the $\vec{r}$ dependence of $G(\vec{r},\vec{r}')$ as a superposition over spherical harmonics in (θ,φ):

$$G(\vec{r},\vec{r}') = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} G_{lm}(\underbrace{r}_{\text{radial coordinate}}, \underbrace{\vec{r}'}_{\text{vector}})\, Y_{lm}(\theta,\varphi)$$

Put these expressions into $\nabla^2 G(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}')$ using ∇^2 in spherical coordinates and recognizing that

$$\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}\right] Y_{lm}(\theta,\varphi) = -l(l+1)\, Y_{lm}(\theta,\varphi)$$

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} Y_{\ell m}(\theta,\varphi)\left[\frac{1}{r}\frac{d^2}{dr^2}\left(r\, G_{\ell m}(r,\vec{r}\,')\right) - \frac{\ell(\ell+1)}{r^2\ \text{[redacted]}}\, G_{\ell m}(r,\vec{r}\,')\right] =$$

$$= \frac{1}{r'^2}\,\delta(r-r')\sum_{\tilde{\ell}}\sum_{\tilde{m}=-\tilde{\ell}}^{+\tilde{\ell}} Y_{\tilde{\ell}\tilde{m}}(\theta,\varphi)\, Y^*_{\tilde{\ell}\tilde{m}}(\theta',\varphi')$$

Multiply and integrate $\int d\Omega\, Y^*_{\bar{\ell}\bar{m}}(\theta,\varphi)$ to pick out $\ell = \bar{\ell} = \tilde{\ell}$, $m = \bar{m} = \tilde{m}$, leaving

$$\frac{1}{r}\frac{d^2}{dr^2}\left(r\, G_{\ell m}(r,\vec{r}\,')\right) - \frac{\ell(\ell+1)}{r^2\ \text{[redacted]}}\, G_{\ell m}(r,\vec{r}\,') = \frac{\delta(r-r')}{r'^2}\, Y^*_{\ell m}(\theta',\varphi')$$

This must be the angular dependence (θ',φ) of $G_{\ell m}(r,\vec{r}\,')$

$$G_{\ell m}(r,\vec{r}\,') = g_\ell(r,r')\, Y^*_{\ell m}(\theta',\varphi')$$

radial coordinate

Effectively this leaves a 1-d Green's fn problem

$$\frac{1}{r}\frac{d^2}{dr^2}\left(r\, g_\ell(r,r')\right) - \frac{\ell(\ell+1)}{r^2\ \text{[redacted]}}\, g_\ell(r,r') = \frac{\delta(r-r')}{r'^2}$$

So solve as usual for $r \neq r'$ in regions $r<r'$ and $r>r'$. Develop the appropriate b.c.'s and continuity conditions and apply them to fully determine $g_\ell(r,r')$.

For $r \neq r'$, note that

$$\frac{1}{r}\frac{d^2}{dr^2}\left(r g_\ell(r,r')\right) - \frac{\ell(\ell+1)}{r^2} g_\ell(r,r') = 0$$

is just the usual radial part of Laplace's eq. in spherical coordinates. We know the solns

$$g_\ell(r,r') = \begin{cases} A r^\ell + \dfrac{B}{r^{\ell+1}} & r < r' \\[2ex] C r^\ell + \dfrac{D}{r^{\ell+1}} & r > r' \end{cases}$$

As before, take $g_\ell(r,r')$ to be continuous at $r = r'$.

$$g_\ell(r'+\epsilon, r') = g_\ell(r'-\epsilon, r') \qquad \epsilon \to 0$$

The other continuity condition is obtained, as before, by integrating the eqn obeyed by g_ℓ (pg
over $r'-\epsilon \to r'+\epsilon$.

$$\underbrace{\int_{r'-\epsilon}^{r'+\epsilon} dr \left[\frac{d^2}{dr^2}\left(r g_\ell(r,r')\right)\right.}_{\frac{d}{dr}\left(r g_\ell(r,r')\right)\Big|_{r'-\epsilon}^{r'+\epsilon}} \underbrace{\left. - \frac{\ell(\ell+1)}{r} g_\ell(r,r')\right]}_{\text{Assume this} \to 0 \text{ as } \epsilon \to 0} = \frac{1}{(r')^2} \underbrace{\int_{r'-\epsilon}^{r'+\epsilon} dr\, r\,\delta(r-r')}_{r'}$$

$$\therefore \quad \frac{d}{dr}\left(r g_\ell(r,r')\right)\Big|_{r=r'-\epsilon}^{r=r'+\epsilon} = \frac{1}{r'}$$

Sec 8.9

Wyld considers the particular case of the interior of a sphere of radius a — an electrostatics example where the Green's fn is found for the case of a point charge in a grounded sphere. The electrostatic potential obeys Poisson's equation for a pt charge and vanishes on the sphere of radius a. Thus the corresponding boundary conditions on g_ℓ are that

- it is finite at $r=0$
- it is zero at $r=a$

With these boundary conditions, the 4 unknown coefficients A, B, C, D are reduced to 2 — A, C. Applying the two continuity conditions determ these two.

Wyld (Sec 8.8) also solves this problem in a closed form using the standard electrostatics method of images. The series form

$$G(\vec{r},\vec{r}\,') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} g_\ell(r,r')\, Y_{\ell m}(\theta,\varphi)\, Y^*_{\ell m}(\theta',\varphi')$$

spherical harmonics addn thm

$$= \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell+1)\, g_\ell(r,r')\, P_\ell(\cos\theta_{\vec{r}\vec{r}\,'})$$

angle btwn $\vec{r}$'s

can be equated to the closed form.

Wyld (Sections 8.10 and 8.11) also considers an example of the Helmholtz equation.

- uses 2-d example of forced circular drumhead so works with polar coordinates r and θ (extension to 3-d for situation with cylindrical symmetry: use (ρ, φ, z)

- Assume a Green's fn satisfying

$$(\nabla^2 + k^2)\, G(\vec{r}, \vec{r}\,') = \delta(\vec{r} - \vec{r}\,') = \frac{1}{r}\delta(r-r')\delta(\theta-\theta')$$

- Express the δ function in the angle as an expansion over Fourier fns (just the expression of closure (completeness))

$$\delta(\theta-\theta') = \frac{1}{2\pi}\sum_{m=-\infty}^{+\infty} e^{im(\theta-\theta')}$$

- Assume an expansion of the Green's fn

$$G(\vec{r}, \vec{r}\,') = \sum_{m=-\infty}^{+\infty} G_m(\vec{r}, \vec{r}\,')\, e^{im\theta}$$

- Use these forms for $\delta(\theta-\theta')$ and G in the eqn for G; recognize the θ' dependence and reduce the problem to finding a 1-d Green's fn that satisfies

$$\frac{d^2 g_m(r,r')}{dr^2} + \frac{1}{r}\frac{d g_m(r,r')}{dr} + \left(k^2 - \frac{m^2}{r^2}\right) g_m(r,r') = \frac{1}{r}\delta(r-r')$$

This is just Bessel's eqn of order m when $r \neq r'$. So $\rightarrow$ usual process

- write soln for $r<r'$ and for $r>r'$
- apply b.c.s
- determine and apply continuity conditions

  integrate eq for g_m over infinitesimal range about $r=r'$

$\rightarrow$ Solution in closed form.

Alternately, express $G(\vec{r},\vec{r}\,') = \sum_n \frac{u_n(\vec{r})\, u_n(\vec{r}\,')}{k^2 - k_n^2}$

where u_n satisfy homo. Helmholtz

$$(\nabla^2 + k_n^2)\, u_n(\vec{r}) = 0$$

In case considered:

$$u_{nm1}(r,\theta) \sim J_m(k_{mn} r) \sin m\theta$$

$$u_{nm2}(r,\theta) \sim \quad " \quad \cos m\theta$$

$$\Sigma \sim \sum \frac{[J \sin J' \sin' + J \cos J' \cos']}{\text{denom}} \sim \sum \frac{J J' \cos m(\theta-\theta')}{\text{denom}}$$

Should also consider a case where the region of interest extends through all space — no surfaces to provide boundary conditions.

Work in spherical coordinates.

Consider Helmholtz eqn example (3-d):

$$(\nabla^2 + k^2)\,\psi(\vec{r}) = -4\pi\rho(\vec{r})$$

(If this comes from wave eqn $k = \omega/c$)

Assume a Green's fn satisfies:

$$(\nabla^2 + k^2)\,G(\vec{r},\vec{r}\,') = \delta(\vec{r}-\vec{r}\,')$$

Usual process to express ψ in terms of G
Apply Green's theorem with $u = \psi$ and $v = G$ to the construction

$$\int d^3r \left[\psi(\vec{r})\underbrace{\nabla^2 G(\vec{r},\vec{r}\,')}_{=\delta(\vec{r}-\vec{r}\,') - \cancel{k^2 G}} - G(\vec{r},\vec{r}\,')\underbrace{\nabla^2 \psi(\vec{r})}_{=-4\pi\rho(\vec{r}) - \cancel{k^2\psi}} \right]$$

"extra" k^2 terms cancel so result as before

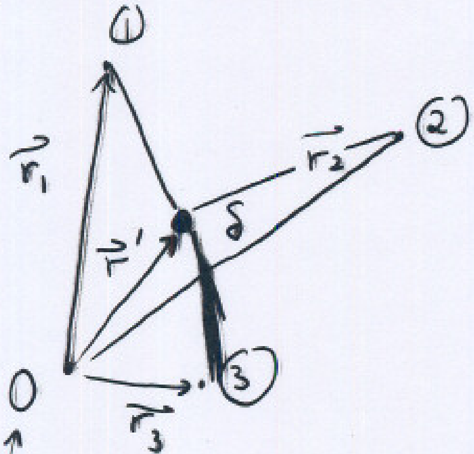
$$\psi(\vec{r}) = -4\pi \int d^3x'\, G(\vec{r},\vec{r}\,')\,\rho(\vec{r}\,') \ + \text{surface terms}$$

(pg 258)

No surfaces to provide b.c.'s $\rightarrow$ one pt in space is the same as the next; no direction singled out:

space – homogeneous + isotropic

Assume $G(\vec{r}, \vec{r}\,')$ assumes only on distance from source $\longrightarrow |\vec{r} - \vec{r}\,'|$.



①, ②, ③ equidistant from δ fn source at $\vec{r}\,'$.

So source has same effect at each of these pts.

$\uparrow$ arbitrarily placed origin of coords. $r_3 < r_1$; $r_3 < r_2$

So $G(\vec{r}, \vec{r}\,')$ can be treated as a 1-d Green's function $G(|\vec{r} - \vec{r}\,'|)$ that satisfies, for $\vec{r} \neq \vec{r}\,'$,

$$(\nabla^2 + k^2)\, G(|\vec{r} - \vec{r}\,'|) = 0$$

The solutions to this equation are just the $\boxed{\text{radial}}$ part of the $\boxed{\text{solutions of}}$ the $\boxed{\text{homogeneous Helmholtz equation}}$ in $\boxed{\text{spherical}}$ coordinates $\longrightarrow$ previously determined as spherical Bessel and Neumann or spherical Hankel fns of first and second kind. Since we have no angular dependence, only the $\boxed{l=0}$ solution are relevant.

If the Helmholtz eqn has arisen as the spatial part of a wave eqn, choose the travelling wave solutions –

$$G_{out}(\vec{r},\vec{r}\,') \sim h_0^{(1)}(k|\vec{r}-\vec{r}\,'|) \sim -\frac{1}{4\pi}\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}$$

$$G_{in}(\vec{r},\vec{r}\,') \sim h_0^{(2)}(k|\vec{r}-\vec{r}\,'|) \sim -\frac{1}{4\pi}\frac{e^{-ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}$$

The "in" and "out" labels correspond to the choice $e^{-i\omega t}$ for the time dependence of outgoing and incoming spherical waves.

Now move on to consider time dependent equations — wave and diffusion – with sources via Green's function method.

$$G(\vec{r},t;\vec{r}\,',t')$$

Wave Equation

Consider a volume V bounded by a surface S. Some quantity $\psi(\vec{r},t)$ obeys the wave equation

$$\nabla^2\psi(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2\psi(\vec{r},t)}{\partial t^2} = -4\pi\rho(\vec{r},t)$$

Either $\psi(\vec{r},t)$ ~~[illegible]~~ itself is given on S

or $\vec{\nabla}\psi(\vec{r},t)\cdot\hat{n}\Big|_S = \dfrac{\partial\psi(\vec{r},t)}{\partial n}$ is given on S

Additionally, initial conditions specified throughout V:

$\psi(\vec{r}, t=T)$ given

$\dfrac{\partial\psi(\vec{r},t)}{\partial t}\Big|_{t=T}$ given

Assume a Green's function satisfying

$$* \quad \nabla^2 G(\vec{r},t;\vec{r}\,',t') - \frac{1}{c^2}\frac{\partial^2 G(\vec{r},t;\vec{r}\,',t')}{\partial t^2} = \delta(\vec{r}-\vec{r}\,')\delta(t-t')$$

G will be subject to the conditions

Either $G(\vec{r},t;\vec{r}\,';t') = 0$ for $\vec{r}$ on S if ψ is given on S

or $\dfrac{\partial G(\vec{r},t;\vec{r}\,';t)}{\partial n}\Big|_S = 0$ for $\dfrac{\partial\psi}{\partial n}$ given on S

$G(\vec{r},t;\vec{r}\,',t') = 0$ for $t < t'$

(source at $\vec{r}\,',t'$)

causality

Symmetry property: $G(\vec{r},t;\vec{r}\,',t') = G(\vec{r}\,',-t';\vec{r},-t)$

As usual we can obtain a formal solution $\psi(\underbrace{\vec{r},t}_{\text{unprimed}})$ in terms of G. To do so, we'll

use eqn for ψ in terms of primed coordinates

$$\nabla'^2 \psi(\vec{r}',t') - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\psi(\vec{r}',t') = -4\pi\rho(\vec{r}',t')$$

and the following for G $[\overset{*}{\vec{r}} \leftrightarrow \vec{r}'; t \leftrightarrow -t'$, symm

$$\nabla'^2 G(\vec{r},t;\vec{r}',t') - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} G(\vec{r},t;\vec{r}',t') = \delta(\vec{r}-\vec{r}')\delta(t-t'$$

Do a similar Green's thm type construction with $u = G$ and $v = \psi$ but include some extra terms to cancel time derivatives. Integrate over the primed coordinates $\vec{r}'$ & t'.

$$I = \int_T^\infty dt' \int_V d^3x' \Big\{ \Big[G(\vec{r},t;\vec{r}',t') \underbrace{\nabla'^2 \psi(\vec{r}',t')}_{= \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\psi(\vec{r}',t') - 4\pi\rho(\vec{r}',t')}$$

$$- \psi(\vec{r}',t') \underbrace{\nabla'^2 G(\vec{r},t;\vec{r}',t')}_{= \frac{1}{c^2}\frac{\partial^2 G}{\partial t'^2}(\vec{r},t;\vec{r}',t') + \delta(\vec{r}-\vec{r}')\delta(t-t')} \Big]$$

$$- \frac{1}{c^2}\Big[G(\vec{r},t;\vec{r}',t') \frac{\partial^2}{\partial t'^2}\psi(\vec{r}',t')$$

$$- \psi(\vec{r}',t') \frac{\partial^2}{\partial t'^2} G(\vec{r},t;\vec{r}',t') \Big] \Big\}$$

$=$ terms cancel

$\approx$ 〃

$\times\times\times\times$ terms survive:

$$I = -4\pi \left\{ \int_T^{\infty} dt' \int_V d^3x' \, G(\vec{r},t;\vec{r}\,',t')\, \rho(\vec{r}\,',t') \right\} - \psi(\vec{r},t) \qquad (1)$$

what we're after

I can be expressed differently by rewriting $[[\sim]]$ as $\dfrac{\partial}{\partial t'}\left[G(\vec{r},t;\vec{r}\,',t') \dfrac{\partial \psi(\vec{r}\,',t')}{\partial t'} - \psi \dfrac{\partial G}{\partial t'} \right]$

so that the integration over t' is easily done and by using Green's thm for the volume integral in the first term.

$$I = \int_T^{\infty} dt' \int_S dA' \left[G(\vec{r},t;\vec{r}\,',t') \frac{\partial \psi(\vec{r}\,',t')}{\partial n'} - \psi(\vec{r}\,',t') \frac{\partial}{\partial n'} G(\vec{r},t;\vec{r}\,',t') \right]$$
$$- \frac{1}{c^2} \int_V d^3x' \left[G(\vec{r},t;\vec{r}\,',t') \frac{\partial \psi(\vec{r}\,',t')}{\partial t'} - \psi(\vec{r}\,',t') \frac{\partial G(\vec{r},t;\vec{r}\,',t')}{\partial t'} \right] \Bigg|_{t'=T}^{t'=\infty} \qquad (2)$$

For $t<t'$, $G(\vec{r},t;\vec{r}\,',t')=0$ so the t' integrals need only cover $T<t'<t$. By the same reasoning, for finite t, the $t'=\infty$ limit in the last term in (2) vanishes.

Equating the two expressions (1) and (2) yields —

$$\psi(\vec{r},t) = -4\pi \int_T^t dt' \int_V d^3x' \; G(\vec{r},t;\vec{r}',t')\,\rho(\vec{r}',t')$$

$$- \int_T^t dt' \int_S dA' \left[G \frac{\partial \psi}{\partial n'} - \psi \frac{\partial G}{\partial n'} \right]$$

- one of these vanishes on S due to b.c.
- other term

$$- \frac{1}{c^2} \int_V d^3x' \left[G \left.\frac{\partial \psi}{\partial t'}\right|_{t'=T} - \psi \left.\frac{\partial G}{\partial t'}\right|_{t'=T} \right]$$

- determined by ψ initial conditions

determined by ψ b.c.

This represents a formal solution for ψ, given G.

Now specialize to the case that V extends through all space $(S \longrightarrow \infty)$ to determine a form for the time dependent Green's function.

Again $G(\vec{r},t;\vec{r}',t') \longrightarrow G(|\vec{r}-\vec{r}'|\,;\,t-t')$

$$\underbrace{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)}_{\Box} G(|\vec{r}-\vec{r}'|,\,t-t') = \delta(\vec{r}-\vec{r}')\,\delta(t-t')$$

$\Box$ d'Alembertian

Temporarily put $\vec{r}'=0$, $t'=0$ to save writing. Restore later

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right] G(\vec{r},t) = \delta(\vec{r})\ \delta(t) \qquad *$$

$$G(\vec{r},t) = 0 \quad \text{for} \quad t<0$$

Solve for $G(\vec{r},t)$ using yet another method: Fourier transform.

Express the spatial δ fn as a Fourier transform

$$\delta(\vec{r}) = \frac{1}{(2\pi)^3}\int d^3k\ e^{i\vec{k}\cdot\vec{r}}$$

Express $G(\vec{r},t)$ formally in terms of a Fourier transform in the spatial coordinates.

$$G(\vec{r},t) = \frac{1}{(2\pi)^3}\int d^3k\ \ e^{i\vec{k}\cdot\vec{r}}\,\tilde{G}(\vec{k},t)$$

where $\tilde{G}(\vec{k},t) = \int d^3x\ e^{-i\vec{k}\cdot\vec{r}}\,G(\vec{r},t)$

Putting these expressions into $*$ above yields an equation for $\tilde{G}(\vec{k},t)$

$$\frac{1}{(2\pi)^3}\int d^3k \left[-k^2 e^{i\vec{k}\cdot\vec{r}}\tilde{G}(\vec{k},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\tilde{G}(\vec{k},t) \right] = \delta(t)\frac{1}{(2\pi)^3}\int d^3k\, e^{i\vec{k}\cdot\vec{r}}$$

Equating the integrands

$$\left(-k^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right)\tilde{G}(\vec{k},t) = \delta(t)$$

This is like an equation for a 1-d Green's fn (in the variable t, time). So solve for $t<0$ and $t>0$.

$t<0$: $\quad \tilde{G}(\vec{k},t)$ must be 0 so that $G(\vec{r},t<0)=0$

$t>0$: $\quad \tilde{G}(\vec{k},t) = A e^{ikct} + B e^{-ikct}$

Also impose continuity conditions about $t=0$.

$$\tilde{G}(\vec{k},0-\epsilon) = \tilde{G}(\vec{k},0+\epsilon) \qquad (1)$$

and
$$\int_{-\epsilon}^{+\epsilon} dt \left(-k^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\tilde{G}(\vec{k},t) = \int_{-\epsilon}^{+\epsilon} dt\ \delta(t) = 1$$

$$\Longrightarrow \quad \frac{\partial \tilde{G}(\vec{k},t)}{\partial t}\Bigg|_{t=-\epsilon}^{t=+\epsilon} = -c^2 \qquad (2)$$

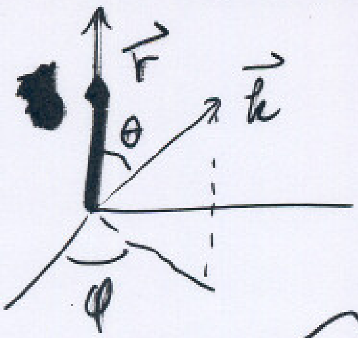
Applying these continuity conditions to the piecewise continuous solns above –

(1) $\quad 0 = Ae^{i\epsilon} + Be^{-i\epsilon} = A + B \quad \rightarrow \quad B = -A$

(2) $\quad \left[Aikce^{i\epsilon} - A(-ikc)e^{-i\epsilon}\right] - 0 = -c^2$

$$\Rightarrow A = -c/2ik$$

$$\therefore \tilde{G}(\vec{k},t) = \begin{cases} -\dfrac{c}{2ik}\left[e^{ikct} - e^{-ikct}\right] & t>0 \\ 0 & t<0 \end{cases}$$



θ = angle btwn $\vec k$ and $\vec r$

$$G(\vec r, t) = \frac{1}{(2\pi)^3}\int d^3k\, \overbrace{\left[e^{ikr\cos\theta}\right]}^{e^{i\vec k\cdot\vec r}} \left(-\frac{c}{2ik}\right)\left[e^{ikct} - e^{-ikct}\right]$$

$t>$

$$\int d^3k = \int_0^\infty k^2\, dk \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \int_{-1}^{+1} d\cos\theta$$

$$\int_{-1}^{+1} d\cos\theta\, e^{ikr\cos\theta} = \frac{1}{ikr}\left[e^{ikr} - e^{-ikr}\right]$$

$$G(\vec r, t) = \frac{1}{(2\pi)^2}\left(-\frac{c}{2i}\right)\left(\frac{1}{ir}\right)\int_0^\infty dk\left[e^{ik(r+ct)} - e^{ik(r-ct)} - e^{-ik(r-ct)} + e^{-ik(r+ct)}\right]$$

$t>$

$$= \frac{c}{8\pi^2 r}\underbrace{\int_{-\infty}^{+\infty} dk\left[e^{ik(r+ct)} - e^{ik(r-ct)}\right]}_{=(2\pi)\left[\delta(r+ct) - \delta(r-ct)\right]} \qquad t>0$$

since $r>0$, this δfn vanishes for $t>0$

$$G(\vec r, t) = \begin{cases} 0 & t<0 \\ -\dfrac{c}{4\pi r}\delta(r-ct) = -\dfrac{1}{4\pi r}\delta\left(\dfrac{r}{c} - t\right) & t>0 \end{cases}$$

Restoring nonzero $\vec{r}'$, t':

$$G(\vec{r},t;\vec{r}',t') = \begin{cases} 0 & \text{for } t<t' \\ -\frac{1}{4\pi}\,\frac{\delta\left[\frac{1}{c}|\vec{r}-\vec{r}'|-(t-t')\right]}{|\vec{r}-\vec{r}'|} & \text{for } t>t' \end{cases}$$

This is the so-called retarded Green's function exhibiting causal behaviour.

G represents the result of a δ fn disturbance at time t' and position $\vec{r}'$. It represents a spherical "wavelet" that travels out from this source. At a time $t>t'$, the wavefront is at $|\vec{r}-\vec{r}'| = c(t-t')$. The amplitude decreases as $\frac{1}{|\vec{r}-\vec{r}'|}$ as time increases.

The complex result for $\psi(\vec{r},t)$ represents the superposition of wavelets due to a source $\rho(\vec{r}',t')$ subject to boundary and initial conditi

For this case $S \longrightarrow \infty$, the result for ψ simplifies somewhat. Put G above into ψ on page 276.

$$\psi(\vec r,t) = -4\pi \int\limits_{\substack{V \\ |\vec r-\vec r'|\le c(t-T)}} d^3x' \left(-\frac{1}{4\pi}\right) \frac{\left[\rho(\vec r,t')\right]_{\text{retarded}}}{|\vec r-\vec r'|}$$

evaluated at $t' = t - \frac{1}{c}|\vec r-\vec r'|$

$$-\int_T^t dt' \underbrace{\int_S dA' \left[G(\vec r,t;\vec r',t') \frac{\partial \psi}{\partial n'}(\vec r',t') - \psi(\vec r',t')\frac{\partial G}{\partial n'} \right]}_{\to\ \phi}$$

$$-\frac{1}{c^2}\int_V d^3x' \left[\left(G(\vec r,t;\vec r',t')\frac{\partial \psi(\vec r',t')}{\partial t'} \right)\Bigg|_{t'=T} - \left(\psi(\vec r',t')\frac{\partial G(\vec r,t;\vec r',t')}{\partial t'} \right)\Bigg|_{t'=T} \right]$$

Second integral: Evaluated on S where $\vec r' \to \infty$. But G sets $|\vec r-\vec r'| = c(t-t')$ with t' lying in the range $T \to t$. G represents a disturbance at $\vec r', t'$. The observation point is at $\vec r, t \to$ finite. The effect of the disturbance at $\vec r' = \infty$ can't propagate to finite $\vec r$ at time t.

$$\delta\left[\,|\vec r-\vec r'| - c(t-t')\right] \text{ vanishes, for } \vec r' \to \infty.$$

So the surface integral vanishes.

Third integral: For simplicity, temporarily choose $\vec r$ at the origin of coordinates and then restore it later. Use spherical coordinates for the integration. The integrand is evaluated at the initial time $t=T$ so denote $\psi(\vec r,T) = F(\vec r)$ and $\frac{\partial \psi(\vec r,t)}{\partial t}\Big|_{t=T} = D(\vec r)$

The third integral becomes

$$-\frac{1}{c^2}\int d\Omega' \int_0^\infty dr'\, r'^2 \Big[-\frac{1}{4\pi} \frac{\delta(t-T-\frac{r'}{c})}{r'} D(r',\Omega')$$

$$\underbrace{\delta(t-T-\tfrac{r'}{c})}_{= c\,\delta(r'-ct+cT)}$$

$$- F(r',\Omega')\left(\frac{-1}{4\pi}\right)\frac{1}{r'}\frac{\partial}{\partial t'}\delta(t'+\frac{r'}{c}-t)\Big|_{t'=T}\Big]$$

Use $\frac{\partial}{\partial t'}\delta(t'+\frac{r'}{c}-t) = c^2 \frac{\partial}{\partial r'}\delta(r'-ct+ct')$

Evaluate the first r' integral using the δ fn. and the second using $\int dx\, f(x)\,\delta'(x-a) = -f'(a)$.

$$\frac{1}{4\pi}\int d\Omega' \Big[(t-T)\, D\big(c(t-T),\Omega'\big) + \frac{\partial}{\partial r'}\big(r' F(r',\Omega')\big)\Big|_{r'=c(t-T)} \Big]$$

$$= \frac{1}{4\pi}\int d\Omega' \Big[(t-T)\frac{\partial \psi(\vec{r}',t')}{\partial t'}\Big|_{\substack{t'=T \\ |\vec{r}-\vec{r}'|=c(t-T)}} + \frac{\partial}{\partial |\vec{r}-\vec{r}'|}\big[|\vec{r}'-\vec{r}|\, \psi(\vec{r}',T)\big]\Big]\Big|_{|\vec{r}'-\vec{r}|=c(t-T)}$$

We have $\vec{r}' - \underset{\substack{\uparrow \\ \text{finite}}}{\vec{r}} = c(t-T)$

Going to the limit initial time $T \to -\infty$, this implies $\vec{r}' \to \infty$ so the integral is evaluated on the surface of an infinitely large sphere in the distant past. Assuming $\psi(\vec{r}',t')$ vanishes for $\vec{r}'\to\infty,\ t'\to-\infty$, $\underset{\to 0}{\text{integral}}$

This leaves only one term for $\psi(\vec{r},t)$.

$$\psi(\vec{r},t) = \int_{|\vec{r}-\vec{r}'| \le c(t-T)} d^3x' \; \frac{\rho\left(\vec{r}', \; t - \frac{1}{c}|\vec{r}-\vec{r}'|\right)}{|\vec{r}-\vec{r}'|}$$

This is the so-called retarded potential solution of the wave equation.

Instead of entirely dumping the last surface integral, can recognize it as due to incoming waves generated by an infinitely remote source outside the region of interest for the source ρ.

$$\psi(\vec{r},t) = \psi_{in}(\vec{r},t) + \text{term above}$$

solution of the homogeneous wave eq.

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi_{in}(\vec{r},t) = 0$$

[Wyld (Sec 8.16) applies the retarded Green's fn to find the field due to a point source moving with $v < c$ and $v > c$ (Cerenkov radiation)]

Diffusion Equation

Repeat the process! $\qquad$ V bounded by S

$$\nabla^2 \psi(\vec{r},t) - \frac{1}{K}\frac{\partial \psi(\vec{r},t)}{\partial t} = -4\pi\rho(\vec{r},t)$$

with <u>either</u> $\psi(\vec{r},t)$ given on S
<u>or</u> $\left.\frac{\partial \psi(\vec{r},t)}{\partial n}\right|_S$ given for $\vec{r}$ on S] b.c.

and $\left.\psi(\vec{r},t)\right|_{t=T}$ given throughout V] initial condition

Assume $\nabla^2 G(\vec{r},t;\vec{r}\,',t') - \frac{1}{K}\frac{\partial G}{\partial t} = \delta(\vec{r}-\vec{r}\,')\delta(t-t')$

with <u>either</u> $G(\vec{r},t;\vec{r}\,',t') = 0$ for $\vec{r}$ on S

<u>or</u> $\frac{\partial G}{\partial n}(\vec{r},t;\vec{r}\,',t') = 0$ for $\vec{r}$ on S

and $G(\vec{r},t;\vec{r}\,',t') = 0$ for $t < t'$.

As before $G(\vec{r},t;\vec{r}\,',t') = G(\vec{r}\,',-t';\vec{r},-t)$

In a way similar to the wave equation case, obtain a formal solution for $\psi(\vec{r},t)$ in terms of G.

Result:

$$\psi(\vec{r},t) = -4\pi \int_T^t dt' \int_V d^3x' \; G(\vec{r},t;\vec{r}',t')\,\rho(\vec{r}',t')$$

$$- \int_T^t dt' \int_S dA' \left[G \frac{\partial \psi}{\partial n'} - \psi \frac{\partial G}{\partial n'} \right] \longrightarrow$$ one term or the other will vanish, depending on b.c. on G, other determined by b.c. on ψ

$$- \frac{1}{\kappa} \int_V d^3x' \; G(\vec{r},t;\vec{r}',T)\,\psi(\vec{r}',T) \longrightarrow$$ determined by initial condition on ψ.

Again consider the infinite space case ($S \to \infty$) to find an example of the Green's function for the diffusion equation.

Again $G(\vec{r},t;\vec{r}',t') = G(\vec{r}-\vec{r}',\, t-t')$

Set $\vec{r}'=0$ and $t'=0$ temporarily for convenience.

$$\nabla^2 G(\vec{r},t) - \frac{1}{\kappa}\frac{\partial G(\vec{r},t)}{\partial t} = \delta(\vec{r})\,\delta(t)$$

with $G(\vec{r},t) = 0$ for $t < 0$.

Solve via Fourier transform in spatial coordinates.

$$G(\vec{r},t) = \frac{1}{(2\pi)^3}\int d^3k \; e^{i\vec{k}\cdot\vec{r}}\,\tilde{G}(\vec{k},t)$$

with $\tilde{G}(\vec{k},t) = \int d^3r \; e^{-i\vec{k}\cdot\vec{r}}\, G(\vec{r},t)$

Express $\delta(\vec{r}) = \frac{1}{(2\pi)^3}\int d^3k \; e^{i\vec{k}\cdot\vec{r}}$.

These yield the equation for $\tilde{G}(\vec{k},t)$ –

$$-k^2\,\tilde{G}(\vec{k},t) - \frac{1}{K}\frac{\partial\tilde{G}}{\partial t}(\vec{k},t) = \delta(t)$$

Solving, for $t<0$ $\quad \tilde{G}(\vec{k},t) = 0$

$t>0$ $\quad \tilde{G}(\vec{k},t) = A e^{-Kk^2 t}$

Integrating the equation for $\tilde{G}$ over an infinitesma range about $t=0$ yields the condition

$$\tilde{G}(\vec{k},+\epsilon) - \tilde{G}(\vec{k},-\epsilon) = -K$$

Applying this condition to the solution determines $A=-K$

$$\therefore \quad \tilde{G}(\vec{k},t) = \begin{cases} 0 & \text{for } t<0 \\ -Ke^{-Kk^2 t} & \text{for } t>0 \end{cases}$$

$$G(\vec{r},t) = \frac{-K}{(2\pi)^3}\int d^3k \; e^{i\vec{k}\cdot\vec{r} - Kk^2 t} \qquad t>0$$

Complete the square and use $\int_{-\infty}^{+\infty} du \; e^{-u^2} = \sqrt{\pi}$

$$G(\vec{r},t) = \frac{-K}{(2\pi)^3}\, e^{-r^2/4Kt} \underbrace{\int \overset{dk_1\,dk_2\,dk_3}{d^3k}\; e^{-Kt\left(\vec{k}-\frac{i\vec{r}}{2Kt}\right)^2}}_{\left(\frac{1}{\sqrt{Kt}}\int_{-\infty}^{+\infty} du\, e^{-u^2}\right)^3 = \left(\frac{\pi}{Kt}\right)^{3/2}} \qquad t>0$$

$$= \frac{-K\, e^{-r^2/4Kt}}{(4\pi K t)^{3/2}}$$

Restoring nonzero $\vec{r}'$, t':

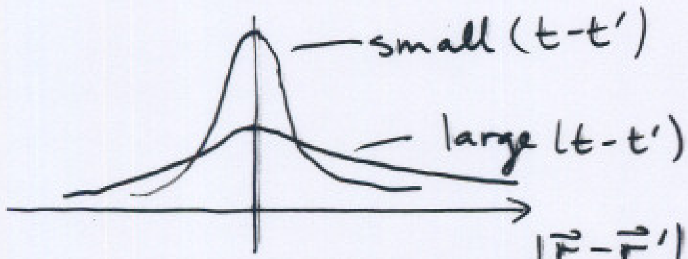
$$G(\vec{r},t;\vec{r}',t') = \begin{cases} 0 & t<t' \\ \dfrac{-K}{[4\pi K(t-t')]^{3/2}} \exp - \left[\dfrac{|\vec{r}-\vec{r}'|^2}{4K(t-t')}\right] & t>t' \end{cases}$$

$e^{-r'^2}$ as $r'\to\infty$

$4K(t-t') \sim \sigma^2$, width in Gaussian distbn

width $\sim [K(t-t')]^{1/2} \longrightarrow$ increases with t

amplitude $\sim [K(t-t')]^{-3/2} \longrightarrow$ decreases with t



$$\int d^3x\; G(\vec{r},t;\vec{r}',t') = -K$$

constant

$$\psi(\vec{r},t) = \frac{1}{\sqrt{4\pi K}} \int_T^t dt' \int_V d^3x' \frac{1}{(t-t')^{3/2}} \exp -\left[\frac{|\vec{r}-\vec{r}'|^2}{4K(t-t')}\right] \rho(\vec{r}',t')$$

$$+ \frac{1}{[4\pi K(t-T)]^{3/2}} \int_V d^3x' \exp-\left[\frac{|\vec{r}-\vec{r}'|^2}{4K(t-t')}\right] \psi(\vec{r}',T)$$

($\int_S$ dropped due to $e^{-r'^2}$; $r'\to\infty$)