Problems with Spherical Symmetry

We established the form of the Laplacian in spherical coordinates. For both the diffusion and wave equations, the spatial dependence reduces to the Helmholtz equation.

$$\nabla^2 u + k^2 u = 0$$

In spherical coords, we saw this separated to the radial and angular dependences as $u(\vec{r}) = R(r)\,Y(\theta,\varphi)$ with

$$\frac{1}{r}\frac{d^2}{dr^2}(rR) + k^2 R - \frac{\lambda}{r^2} R = 0$$

and $$\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right] = -\lambda Y .$$

We already briefly sketched the nature of the solutions to these equations (pg 40-43).

Because we will find that we have to impose a condition on the separation constant λ based on the requirement that our solutions are well behaved at $\cos\theta = \pm 1$, it will be useful to investigate the angular dependence in some detail now. We will then return to the radial dependence with the required form of λ inserted.

Problems with spherical symmetry frequently arise when considering electrostatics or gravitational problems. There we are typically looking at Laplace's or Poisson's equations.

Another important application is to atomic and subatomic systems. In these important quantum systems, the solutions to the angular part of our problems gives us their angular momentum properties. (orbital)

+ electromagnetic radiation problems, scattering....

Separate the polar (θ) and azimuthal (φ) dependances.

$$Y = P(\theta)\,\Phi(\varphi)$$

$$\underbrace{\frac{\sin\theta}{P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \lambda\sin^2\theta}_{\theta \text{ only} = -m^2} + \underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}}_{\varphi \text{ only} = m^2} = 0$$

$$\therefore \frac{d^2\Phi}{d\varphi^2} = -m^2\Phi$$

(m^2, separation constant)

$$\Phi(\varphi) = A_m e^{im\varphi} + B_m e^{-im\varphi} \qquad m = 0, 1, 2, \ldots$$

Demanding $\Phi(\varphi + 2\pi) = \Phi(\varphi)$ implies m integer.

So the problem at hand is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2\theta}P + \lambda P = 0$$

Conventionally we re express this in new variable.

$$\mu \equiv \cos\theta \qquad \therefore \sin\theta = \sqrt{1-\mu^2}$$

$$\frac{d}{d\theta} = \frac{d\mu}{d\theta}\frac{d}{d\mu} = -\sin\theta\frac{d}{d\mu} = -\sqrt{1-\mu^2}\frac{d}{d\mu}$$

The equation becomes:

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right) - \frac{m^2}{(1-\mu^2)}P + \lambda P = 0$$

Associated Legendre Equation

Note that this is S-L form

$$\frac{d}{d\mu}\left(p(\mu)\frac{dP}{d\mu}\right) - q(\mu)P(\mu) + \lambda w(\mu)P(\mu) = 0$$

$$p(\mu) = 1-\mu^2 \qquad q(\mu) = \frac{m^2}{1-\mu^2} \qquad w(\mu) = 1$$

Since $\mu = \cos\theta$, the interval for S-L problem is $-1 \rightarrow +1$
So again we have an eigenfn / eigenvalue problem. We should expect to find a set of orthogonal fns as solutions and to be able to express soln to a particular problem as expansion in these eigenfns.

We can identify the singular points of the equation by rewriting it as:

$$\frac{d^2P}{d\mu^2} - \frac{2\mu}{(1-\mu^2)}\frac{dP}{d\mu} + \frac{1}{(1-\mu^2)}\left[\lambda - \frac{m^2}{(1-\mu^2)}\right]P = 0$$

The points $\mu = \pm 1$ are both regular singular pts. Since $\mu = \cos\theta$, these correspond to the "North and South poles" in our spherical coordinate system.

Generally we will want our solutions to be finite, including at $\cos\theta = \pm 1$. Imposing this boundedness condition will replace other types of B-C boundary conditions. In terms of our series solutions, the requirement that our solns be finite at $\mu = \pm 1$ will imply that the series has to terminate. This will only occur for certain values of λ.

It is conventional to proceed with the simpler $m = 0$ case first. The equation reduces to Legendre's equation.

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right) + \lambda P = 0$$

This is the case of azimuthal symmetry. (Φ indpt of ϕ) (Symmetric under rotation about z-axis.)

The point $\mu=0$ is a regular point so use a series solution:

$$P = \sum_{n=0}^{\infty} a_n \mu^n$$

$$\sum_{n=0}^{\infty} a_n \frac{d}{d\mu}\left((1-\mu^2)\frac{d\mu^n}{d\mu}\right) + \lambda \sum_{n=0}^{\infty} a_n \mu^n = 0$$

Differentiating,

$$\underbrace{\sum_{n=0}^{\infty} a_n n(n-1)\mu^{n-2}}_{\text{Put } n=p+2} + \underbrace{\sum_{n=0}^{\infty} a_n \left[\lambda - n(n+1)\right]\mu^n}_{\text{Put } n=p} = 0$$

We get no info from the powers μ^{-2} and μ^{-1}.

$$\sum_{p=0}^{\infty} \left[a_{p+2}(p+2)(p+1) + a_p\left[\lambda - p(p+1)\right]\right]\mu^p = 0$$

Since each coefficient must vanish, our recursion relation is:

$$a_{p+2} = -\frac{\left[\lambda - p(p+1)\right]}{(p+2)(p+1)} a_p$$

So this generates two series solutions, one in even powers of μ and one in odd powers. We won't prove it but these series diverge for $\mu = \pm 1$ unless they terminate.

We want finite solutions at $\mu = \pm 1$ so we terminate the series by imposing the condition on the separation constant that:

$$\lambda = p(p+1) \qquad p = 0, 1, 2, \ldots$$

(positive integers)

It is conventional to write $\lambda = \ell(\ell+1)$. We label the solutions $P_\ell(\mu)$ and write Legendre's equation as

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP_\ell}{d\mu}\right) + \ell(\ell+1)P_\ell = 0$$

The $P_\ell(\mu)$ are the Legendre polynomials.

By convention, the coefficients a_0, a_1 are chosen such that

$$\boxed{P_\ell(1) = 1}$$

The first few Legendre polynomials are easily obtained.

$\ell = 0$ $\quad P_0(\mu) = a_0\mu^0 \quad P_0(1) = 1 \Rightarrow a_0 = 1$

$$\therefore \boxed{P_0(\mu) = 1}$$

$\ell = 1$ $\quad P_1(\mu) = a_1\mu \quad P_1(1) = 1 \Rightarrow a_1 = 1$

$$\therefore \boxed{P_1(\mu) = \mu}$$

$$a_{p+2} = -\frac{[l(l+1) - p(p+1)]}{(p+2)(p+1)} a_p$$

$l = 2$:

$$a_2 = -\frac{2(3)}{(2)(1)} a_0 = -3a_0$$

$$P_2(\mu) = a_0(1 - 3\mu^2) \qquad P_2(1) = 1 \Rightarrow a_0 = -\frac{1}{2}$$

$$\therefore \boxed{P_2(\mu) = \frac{1}{2}(3\mu^2 - 1)}$$

$l = 3$:

$$a_3 = -\frac{[3(4) - 1(2)]}{(3)(2)} a_1 = -\frac{5}{3} a_1$$

$$P_3(\mu) = a_1\left(\mu - \frac{5}{3}\mu^3\right) \qquad P_3(1) = 1 \Rightarrow a_1 = -\frac{3}{2}$$

$$\therefore \quad P_3(\mu) = \frac{1}{2}(5\mu^3 - 3\mu)$$

etc.

Comment on Legendre functions of second Kind:

Solving Legendre eq, we got two series solutions, one beginning with a_0 (even powers) and one beginning with a_1 (odd powers). Taking $\lambda = l(l+1)$ with l an integer made one of these series terminate and, hence, gave a finite result P_l. The other solution is usually denoted as Q_l. We won't dwell on it because it is not finite at $\mu = \pm 1$.

We can look at one example, $\ell = 0$.

$$a_{p+2} = \frac{p(p+1)}{(p+2)(p+1)} a_p = \left(\frac{p}{p+2}\right) a_p = \left(\frac{p}{p+2}\right) \frac{(p-2)(p-1)}{p(p-1)} a_{p-2}$$

$$= \frac{(p-2)}{(p+2)} a_{p-2} = \cdots\cdots = \frac{a_1}{p+2}$$

$$Q_0(\mu) = a_1 \left(\mu + \frac{\mu^3}{3} + \frac{\mu^5}{5} + \cdots\right)$$

$$= \frac{a_1}{2} \ln\left(\frac{1+\mu}{1-\mu}\right)$$

Conventional to put $a_1 = 1$.

So this is badly behaved at $\mu = \pm 1$.

However, for large $\mu \to \infty$ (arising in a different context), the $Q_\ell(x)$ have a good limiting behaviour:

$$Q_\ell(\mu) \xrightarrow[\mu \to \infty]{} \sqrt{\pi} \frac{\Gamma(\ell+1)}{\Gamma(\ell+3/2)} \frac{1}{(2\mu)^{\ell+1}}$$

(This can be shown by writing the Legendre eqn in terms of a variable $u = \frac{1}{\mu}$ and expanding soln as a series in powers of u.)

Because we have Legendre eqn in S-L form with the b.c. at the endpoints $\mu = \pm 1$ being that the solutions P_ℓ are finite, we expect an orthogonality relation

$$\int_{-1}^{1} d\mu \, P_\ell(\mu) \, P_{\ell'}(\mu) = 0 \quad \text{for} \quad \ell \neq \ell'$$

We can check that this arises in the usual way:

$$\boxed{(P_n, \mathcal{L} P_\ell) - (P_\ell, \mathcal{L} P_n)}$$

where $\mathcal{L} P_\ell = \frac{d}{d\mu}\left((1-\mu^2)\frac{d}{d\mu} P_\ell\right) = ((1-\mu^2)P_\ell')' = -\ell(\ell+1) P_\ell$

$$\text{LHS} = \int_{-1}^{1} d\mu \, P_n ((1-\mu^2)P_\ell')' - \int_{-1}^{1} d\mu \, P_\ell ((1-\mu^2)^2 P_n')' = [-\ell(\ell+1) + n(n+1)] \times \int_{-1}^{1} d\mu \, P_n P_\ell$$

Integrating by parts:

$$\text{LHS} = \left[P_n \cancel{(1-\mu^2)} P_\ell'\right]_{-1}^{1} - \int_{-1}^{1} d\mu \, \cancel{P_n' (1-\mu^2) P_\ell'}$$

$$- \left\{ \left[P_\ell \cancel{(1-\mu^2)} P_n'\right]_{-1}^{1} - \int d\mu \, \cancel{P_\ell' (1-\mu^2) P_n'} \right\} = 0$$

Thus RHS = 0

$$[\ell(\ell+1) - n(n+1)] \int_{-1}^{1} d\mu\, P_\ell(\mu) P_n(\mu) = 0$$

For $\ell \neq n$, this implies

$$\boxed{\int_{-1}^{1} d\mu\, P_\ell\, P_n = 0 \qquad \ell \neq n}$$

The P_ℓ are orthogonal on the interval $(-1, 1)$ with the weight function $w(\mu) = 1$.

Thus we expect to be able to represent a function $f(\mu)$ as an expansion in these orthogonal eigenfns.

$$f(\mu) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(\mu)$$

In order to be able to determine the coefficients A_ℓ, we need the normalization $\int_{-1}^{1} d\mu\, P_\ell^2 = ?$

This can be done various ways. It is useful to develop a closed form for the P_ℓ known as Rodrigues' formula and also to develop the generating function for the P_ℓ.

We will start by stating Rodrigues' formula:

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell = \frac{(-1)^\ell}{2^\ell \ell!} f_\ell(x)$$

First, show that $f_\ell(x) = \frac{d^\ell}{dx^\ell}(1-x^2)^\ell$, when expressed as a series in powers of x, satisfies the recursion relation for the P_ℓ.

Recall binomial theorem:

$$(1+z)^\ell = \sum_{k=0}^{\ell} \binom{\ell}{k} z^\ell \qquad \text{where } \binom{\ell}{k} = \frac{\ell!}{k!(\ell-k)!}$$

Apply this for $z = -x^2$.

$$(1-x^2)^\ell = \sum_{k=0}^{\ell} \frac{\ell!}{k!(\ell-k)!} (-1)^k x^{2k}$$

For f_ℓ we need the ℓth derivative of x^{2k}, which vanishes for $\ell > 2k$.

For $\ell \le 2k$, $\frac{d^\ell}{dx^\ell} x^{2k} = 2k(2k-1)\cdots(2k-(\ell+1))\, x^{2k-\ell}$

$$= \frac{(2k)!}{(2k-\ell)!} x^{2k-\ell}$$

Putting this together,

$$f_\ell = \frac{d^\ell}{dx^\ell}(1-x^2)^\ell = \sum_{k \geq \ell/2}^{\ell} \frac{\ell!}{k!\,(\ell-k)!}(-1)^k \left[\frac{(2k)!}{(2k-\ell)!}\, x^{2k-\ell}\right]$$

So $$P_\ell = \frac{(-1)^\ell}{2^\ell \ell!} \sum_{k \geq \ell/2}^{\ell} \frac{(-1)^k\, \ell!\, (2k)!}{k!\,(\ell-k)!\,(2k-\ell)!}\, x^{2k-\ell} \qquad *$$

Note

$$= \sum_{n=0}^{\infty} a_n x^n \quad \leftarrow \text{our previous starting point}$$

$P_\ell(1) = 1$

We want to check the recursion relation for the coefficients in * against our previous result.

For a_n, set $2k-\ell = n$:

$$a_n = \frac{(-1)^\ell}{2^\ell \ell!} \frac{(-1)^{(n+\ell)/2}\, (n+\ell)!\, \ell!}{\left(\frac{(n+\ell)}{2}\right)!\left(\frac{(\ell-n)}{2}\right)!\, n!} \quad \times\times\times$$

For a_{n+2}, set $2k-\ell = n+2$:

$$a_{n+2} = \frac{(-1)^\ell}{2^\ell \ell!} \frac{(-1)^{(n+\ell+2)/2}\, (n+\ell+2)!\, \ell!}{\left(\frac{(n+\ell+2)}{2}\right)!\left(\frac{(\ell-n-2)}{2}\right)!\, (n+2)!} \quad \times\times\times$$

(Note: different form will appear on pg 159. Get it by changing dummy index k to $n-k$.)

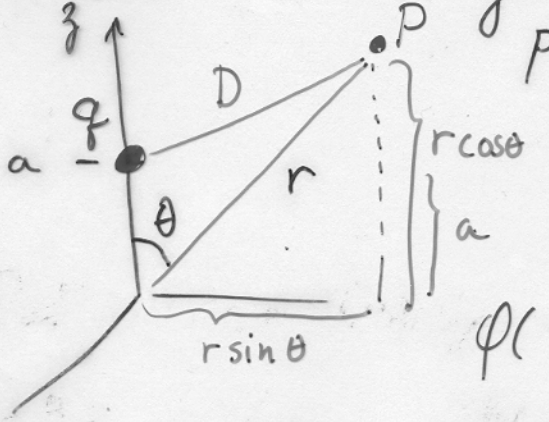
Taking the ratio of coefficients of successive terms,

$$\frac{a_{n+2}}{a_n} = \frac{(-1)[\ell(\ell+1) - n(n+1)]}{(n+2)(n+1)}$$

This is the recursion relation for the Legendre polynomials, as desired. To assess that the normalization in Rodrigues' formula is correct (i.e. give $P_\ell(1)=1$) we could check a few terms. Instead let's get a generating function expression.

In fact, outside of the context of Legendre polynomials as solutions of the Legendre differential equation, this is sometimes how they are defined. We will see the P_ℓ arising from a generating fn in the context of the physical problem of the electrostatic potential due to a point charge.

Consider a charge q placed a distance $z=a$ from the origin on the z-axis. Consider the potential due to the charge at a point P, located a distance r from the origin. The potential at P is:



picture not to scale

$r \gg a$

$$\phi(P) = \frac{q}{4\pi\epsilon_0 D}$$

where $D^2 = (r\sin\theta)^2 + (r\cos\theta - a)^2 = r^2 + a^2 - 2ar\cos\theta$

Thus $\varphi(r) = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}}$

$\mu = \cos\theta$

Put $t \equiv \frac{a}{r}$. Assume the point P is far from the charge q so that t is small.

$$\varphi(r) = \frac{q}{4\pi\epsilon_0 r} \frac{1}{\sqrt{1-2t\mu+t^2}} = \frac{q}{4\pi\epsilon_0 r} \sum_{l=0}^{\infty} P_l(\mu) t^l.$$

express φ as a power series expansion in t and identify the coefficients as the P_l.

On the LHS use the binomial expansion for

$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!} z^2 + \cdots$$

with $z = -2t\mu + t^2$ and $\alpha = -\frac{1}{2}$:

$$(1+z)^{-1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^{2n} (n!)^2} z^n$$

Putting $z = (-2t\mu + t^2)$ and writing $z^n = t^n(-t+2\mu)^n(-1)^n$, then using binomial theorem again in the form $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

Here use $a = 2\mu$ and $b = -t^2$.

So $$[1 - 2t\mu + t^2]^{-1/2} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n (2n)!}{2^{2n} (n!)^2} (-1)^n t^n \binom{n}{k} (2\mu)^{n-k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k (2n)! (2\mu)^{n-k} t^{n+k}}{2^{2n} n! k! (n-k)!} \times (-t)^k$$

Change range and form of summation

$$= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{2^{2n-2k} k! (n-k)! (n-2k)!} (2\mu)^{n-2k} \right\} t^n$$

$*$ indpt of k

$[n/2] = n/2$ for n even and $(n-1)/2$ for n odd

Here we used $$\sum_{n \geq 0} \sum_{k=0}^{n} a(k, n-k) = \sum_{n \geq 0} \sum_{k=0}^{[n/2]} a(k, n-2k)$$

So comparing $[1 - 2t\mu + t^2]^{-1/2} = \sum_{n=0}^{\infty} P_n(\mu) t^n$

we have

$$P_n(\mu) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{2^n k! (n-k)! (n-2k)!} \mu^{n-2k}$$

This is, despite appearances, the same as $*$ on pg 156 from Rodrigues formula.

Now it is easy to show that the $P_\ell(\mu)$ from Rodrigues formula, which are the same as the coefficients in the generating function formula, actually obey the correct (conventional) normalization $P_\ell(1) = 1$.

$$\left[1 - 2t\mu + t^2\right]^{-1/2}\Big|_{\mu=1} = (1-2t+t^2)^{-1/2} = \frac{1}{1-t}$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$\sum_{\ell=0}^{\infty} P_\ell(\mu)\, t^\ell \Big|_{\mu=1} \qquad\qquad 1 + t + t^2 + \cdots + t^\ell + \cdots$$

$$= \sum_{\ell=0}^{\infty} P_\ell(1)\, t^\ell \qquad\qquad = \sum_{\ell=0}^{\infty} t^\ell$$

$$\leftarrow \boxed{P_\ell(1) = 1} \qquad \text{(This also yields } P_\ell(-1) = (-1)^\ell \text{)}$$

Finally, we have various ways of expressing the Legendre functions that allow us to evaluate

$$\int_{-1}^{1} d\mu \left[P_\ell(\mu)\right]^2$$

↑ use Rodrigues' formula here.

$$\int_{-1}^{1} d\mu \left[P_\ell(\mu)\right]^2 = \frac{1}{\left[2^\ell \ell!\right]^2} \int_{-1}^{1} d\mu \, \frac{d^\ell}{d\mu^\ell}(\mu^2-1)^\ell \, \frac{d^\ell}{d\mu^\ell}(\mu^2-1)^\ell$$

Integrate this by parts ℓ times to move the derivatives all over to one of the factors. At each step, the integrated term vanishes when evaluated at the limits, leaving only the next integral.

$$\int_{-1}^{1} d\mu \,[P_\ell(\mu)]^2 = \frac{(-1)^\ell}{[2^\ell \ell!]^2} \int_{-1}^{1} d\mu \,(\mu^2-1)^\ell \frac{d^{2\ell}}{d\mu^{2\ell}} (\mu^2-1)^\ell$$

$(\mu^2-1)^\ell$ is a polynomial that goes like

$\mu^{2\ell} + \cdots\cdots$ ← lower orders in μ. So upo

differentiating 2ℓ times, only the leading term contributes : a factor $(2\ell)!$

$$\int_{-1}^{1} d\mu \,[P_\ell(\mu)]^2 = \frac{(-1)^\ell (2\ell)!}{[2^\ell \ell!]^2} \int_{-1}^{1} d\mu \,(\mu-1)^\ell (\mu+1)^\ell$$

Again integrate ℓ times by parts:

$$= \frac{(-1)^\ell (2\ell)!}{[2^\ell \ell!]^2} \left\{ (\mu-1)^\ell \frac{(\mu+1)^{\ell+1}}{(\ell+1)} \Big|_{-1}^{+1} - \int_{-1}^{+1} d\mu\, \ell (\mu-1)^{\ell-1} \frac{(\mu+1)^{\ell+1}}{(\ell+1)} \right\}$$

vanishes at limits

$$= \frac{(-1)^\ell (2\ell)!}{[2^\ell \ell!]^2} \frac{(-1)^\ell \ell!}{(\ell+1)\cdots(2\ell)} \int_{-1}^{+1} d\mu\, (\mu+1)^{2\ell}$$

$$= \left[(2\ell)!/\ell!\right]$$

$$= \frac{2}{2\ell+1} = \int_{-1}^{1} d\mu \,(P_\ell(\mu))^2$$

$$\int_{-1}^{+1} d\mu\, P_\ell(\mu)\, P_n(\mu) = \frac{2\delta_{\ell n}}{2\ell+1}$$

Now we have our orthonormality relation. So if we expand a function $f(\mu)$ in a set of eigenfunctions, the P_ℓ, as

$$f(\mu) = \sum_\ell a_\ell P_\ell(\mu)$$ Fourier-Legendre series

$$\int_{-1}^{+1} d\mu\, P_n(\mu)$$

we project out the coefficients of the expansion as

$$a_n = \frac{2n+1}{2} \int_{-1}^{+1} d\mu\, P_n(\mu)\, f(\mu)$$ Fourier-Legendre coefficients

Note: it can be useful to use Rodrigues formula to evaluate this type of integral.

Putting a_ℓ (with an integration variable μ') back into the expansion for $f(\mu)$:

$$f(\mu) = \int_{-1}^{+1} d\mu' \left[\sum_\ell \frac{2\ell+1}{2} P_\ell(\mu)\, P_\ell(\mu') \right] f(\mu')$$

allows us to identify

$$\frac{1}{2}\sum_{l}(2l+1)\,P_l(\mu)\,P_l(\mu') = \delta(\mu-\mu')$$

Recursion relations for the Legendre polynomials

Denote the generating function as

$$g(t,\mu) = \frac{1}{[1-2t\mu+t^2]^{1/2}} = \sum_{l=0}^{\infty} P_l(\mu)\,t^l$$

Differentiate g with respect to t:

$$\frac{\partial g(t,\mu)}{\partial t} = \frac{-(t-\mu)}{[1-2t\mu+t^2]^{3/2}} = \frac{(\mu-t)\,g}{[1-2t\mu+t^2]}$$

$$[1-2t\mu+t^2]\frac{\partial g}{\partial t} = (\mu-t)\,g$$

$$[1-2t\mu+t^2]\sum_{l=0}^{\infty} l\,t^{l-1}P_l = (\mu-t)\sum_{l=0}^{\infty} t^l P_l$$

$$\sum_{l=0}^{\infty} t^{l+1}(l+1)P_l - \mu\sum_{l=0}^{\infty} t^l(2l+1)P_l + \sum_{l=0}^{\infty} t^{l-1}\,l\,P_l = 0$$

The coefficient of each power in t must vanish.

$$* \quad \boxed{l P_{l-1}(\mu) - \mu(2l+1) P_l(\mu) + (l+1) P_{l+1}(\mu) = 0} \qquad l \geq 1$$

This is called a "pure recursion relation". It involves the Legendre polynomials with successive values of l and not their derivatives.

Differentiate the generating function with respect to μ:

$$\frac{\partial g(t,\mu)}{\partial \mu} = \frac{t}{[1 - 2t\mu + t^2]^{3/2}} = \frac{tg}{[1 - 2t\mu + t^2]}$$

$$[1 - 2t\mu + t^2] \frac{\partial g}{\partial \mu} = tg$$

Again inserting the series $g = \sum_{l=0}^{\infty} t^l P_l$ and finding coefficients of powers in t:

$$** \quad \boxed{P_l(\mu) = P'_{l-1}(\mu) - 2\mu P'_l(\mu) + P'_{l+1}(\mu)} \qquad l \geq 1$$

Differentiating * wrt μ, can combine it with ** to eliminate either P'_{l+1} or P'_{l-1}:

(a) $$\boxed{l P_l(\mu) = \mu P'_l(\mu) - P'_{l-1}(\mu)} \leftarrow l \geq 1$$

(b) $$\boxed{(l+1) P_l(\mu) = P'_{l+1}(\mu) - \mu P'_l(\mu)} \leftarrow l \geq 0$$

Adding the last two equations gives:

$$(2l+1)\,P_l(\mu) = P'_{l+1}(\mu) - P'_{l-1}(\mu) \qquad l \geq 1$$

Here P_l is expressed in terms of derivatives. This can be useful when evaluating integrals.

We can also find "ladder operators":

Replace l by $(l-1)$ in (b) above:

$$l\,P_{l-1} = P'_l - \mu\,P'_{l-1} = P_l' - \mu\left(\mu P_l' - l P_l\right)$$

(use (a) here)

$$\therefore \quad \boxed{P_{l-1} = \mu P_l + \frac{(1-\mu^2)}{l} P_l'} \qquad l \geq 1$$

This gives us P_{l-1} in terms of P_l so we "step down".

Alternately we can get the step-up ladder operator:

(a) with $l \to l+1$

$$(l+1)\,P_{l+1} = \mu P'_{l+1} - P_l' = \mu\left[(l+1)P_l + \mu P_l'\right] - P_l'$$

(b)

$$\boxed{P_{l+1} = \mu P_l - \frac{(1-\mu^2)}{(l+1)} P_l'}$$

At this point, we are able to solve normal mode problems in a spherical geometry if they are azimuthally symmetric ($m=0$). Let's first proceed to consider the $m \neq 0$ case.

Associated Legendre equation:

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_{lm}^{m}}{d\mu}\right] + \left[l(l+1) - \frac{m^2}{(1-\mu^2)}\right] P_{lm}^{m} = 0$$

This is S-L form

$$\frac{d}{d\mu}\left[p(\mu)\frac{dP_{lm}}{d\mu}\right] - q(\mu)P_{lm} + \lambda w(\mu) P_{lm} = 0$$

Here we will identify $\lambda = l(l+1)$ as our S.L. eigenvalue. Then $w=1$ and we expect

$$\int_{-1}^{1} d\mu\, P_l^m P_{l'}^m = 0 \quad \text{unless} \quad l = l'$$

[We can instead consider this as a S-L problem with eigenvalue m^2. In this case $w = \frac{1}{1-\mu^2}$ and we expect

$$\int_{-1}^{1} d\mu \frac{P_l^m P_l^{m'}}{1-\mu^2} = 0 \quad \text{unless} \quad m = m' \;\Big]$$

We won't do a standard Frobenius series solution for the associated Legendre equation. Instead we will build on our results for the solution of the Legendre equation.

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d}{d\mu}P_\ell(\mu)\right] + \ell(\ell+1)P_\ell(\mu) = 0$$

vs

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d}{d\mu}P_\ell^m(\mu)\right] + \left[\ell(\ell+1) - \frac{m^2}{1-\mu^2}\right]P_\ell^m(\mu) = 0$$

Temporarily adopt a simplified notation:

$P_\ell(\mu) \equiv u \qquad P_\ell^m(\mu) \equiv v \qquad \mu \equiv x$

Denote differentiation with respect to x by primes.

We will find we can express the solutions of the Associated Legendre eq P_ℓ^m in terms of derivatives of solutions of the Legendre equation P_ℓ.

4707/260
3807/168

Legendre eqn.: $(1-x^2)u'' - 2xu' + l(l+1)u = 0$

Associated L. eqn.: $(1-x^2)v'' - 2xv' + \left[l(l+1) - \frac{m^2}{(1-x^2)}\right]v = 0$ *

Given our known solns. u, we can first show that

$$v \equiv (1-x^2)^{m/2} \frac{d^m}{dx^m}(u)$$

for integral $m \geq 0$ satisfies the associated Leg.

Denote $\frac{d^m u}{dx^m} \equiv w \equiv d^m u$

$$v = (1-x^2)^{m/2} w$$

$$v' = (1-x^2)^{m/2} w' - mx(1-x^2)^{\frac{m}{2}-1} w$$

$$v'' = (1-x^2)^{m/2} w'' - 2mx(1-x^2)^{\frac{m}{2}-1} w' - m(1-x^2)^{\frac{m}{2}-1} w + mx^2(m-2)(1-x^2)^{\frac{m}{2}-2} w$$

Put all into * and denote w, w', w'' by $d^m u$, $d^{m+1} u$, $d^{m+2} u$, respectively:

$$\text{LHS of } * = \left[(1-x^2) d^{m+2} u - 2x(m+1) d^{m+1} u + \left(l(l+1) - m(m+1)\right) d^m u\right]$$

4207/261
3807/169

This result can be rewritten in a form where the differentiation d^m is "pulled out". Use Leibnitz's rule for the mth derivative of a product of two fns:

$$d^m(\alpha\beta) = \sum_{n=0}^{m} \binom{m}{n} d^n\alpha\, d^{m-n}\beta$$

$$= \alpha d^m\beta + m\, d\alpha\, d^{m-1}\beta + \frac{m(m-1)}{2!} d^2\alpha\, d^{m-2}\beta$$

$$+ \cdots + \frac{m!}{k!(m-k)!} d^k\alpha\, d^{m-k}\beta + \cdots + (d^m\alpha)\beta$$

In this case set $\alpha = (1-x^2)$ and $\beta = u''$ first. Then

$$\frac{d^m}{dx^m}\left[(1-x^2)u''\right] = \sum_{n=0}^{m} \binom{m}{n} d^n(1-x^2)\, \underbrace{d^{m-n}u''}_{d^{m-n+2}u}$$

$$= (1-x^2)d^{m+2}u - 2x\,m\,d^{m+1}u - m(m-1)d^m u$$

Also $$\frac{d^m}{dx^m}\left[-2xu'\right] = (-2x)d^{m+1}u - 2m\,d^m u$$

LHS of * becomes: but this $=0$ by Legendre eqn, so

$$\frac{d^m}{dx^m}\left[(1-x^2)u'' - 2xu' + l(l+1)u\right] = 0$$

Hence $v = (1-x^2)^{m/2} \frac{d^m u}{dx^m}$ is indeed a soln.

of the associated Legendre eqn.

$$P_l^m(\mu) = (1-\mu^2)^{m/2} \frac{d^m}{d\mu^m} P_l(\mu)$$

$$= \frac{1}{2^l l!} (1-\mu^2)^{m/2} \frac{d^{l+m}}{d\mu^{l+m}} (\mu^2-1)^l \qquad (*)$$

Note m must be $\leq l$. P_l^m vanishes for $m > l$.
These are the associated Legendre functions.
[Factor $(1-\mu^2)^{m/2}$ means they are not polynomials for m odd]

17/2001

The associated L. eq. depends on m only as an (m^2) term so it remains the same for $m \longrightarrow -m$. Hence, the soln. obtained replacing $m \longrightarrow -m$

$$P_l^{-m}(\mu) = \frac{1}{2^l l!} (1-\mu^2)^{-m/2} \frac{d^{l-m}}{d\mu^{l-m}} (\mu^2-1)^l$$

is also a solution. In fact, like P_l^m above it is a physically acceptable soln. The associated L. eq., as a second order diff.

eq. should have 2 solns. The "other" soln. turns out to be unacceptable — it is singular at $\mu = \pm 1$ ($\theta = 0, \pi$). Hence, the solns given above must be equal up to a constant multiplicative factor:

$$P_\ell^{-m}(\mu) = A P_\ell^m(\mu)$$

Use the expressions for these, divide by $(1-\mu^2)^{m/2}$ and set $\mu = 1$:

$$(1-\mu^2)^{-m} \frac{d^{\ell-m}}{d\mu^{\ell-m}} (\mu^2-1)^\ell \Big|_{\mu=1} = A \frac{d^{\ell+m}}{d\mu^{\ell+m}} (\mu^2-1)^\ell \Big|_{\mu=1}$$

$$(-1)^m 2^{\ell-m} \frac{\ell!}{m!} = A\, 2^{\ell-m} \frac{\ell!}{m!} \frac{(\ell+m)!}{(\ell-m)!}$$

$$A = (-1)^m \frac{(\ell-m)!}{(\ell+m)!}$$

$$P_\ell^{-m}(\mu) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\mu)$$

(The unphysical solution has the form $f(\theta) = \ln(\tan\theta/2)$ which is singular for $\theta = 0, \pi$.)

There is another form of P_ℓ^m which is often used:

$$P_\ell^m(\mu) = \frac{(-1)^m}{2^\ell} \frac{(\ell+m)!}{\ell!\,(\ell-m)!} (1-\mu^2)^{-m/2} \frac{d^{\ell-m}}{d\mu^{\ell-m}} (\mu^2-1)^\ell \qquad (**)$$

(get this immediately from $P_\ell^m = A^{-1} P_\ell^{-m}$)

The easiest way to find orthonormality relations is to use this form (**) and form (*) on page 262 in $\int_{-1}^{1} d\mu\, P_\ell^m\, P_{\ell'}^m$. Integrating by parts, find

$$\int_{-1}^{1} d\mu \left(P_\ell^m(\mu)\right)^2 = \frac{2}{(2\ell+1)} \frac{(\ell+m)!}{(\ell-m)!}$$

Also $\int_{-1}^{1} d\mu\, P_\ell^m\, P_{\ell'}^m = 0$ for $\ell \neq \ell'$.

N.B. The m, m' orthogonality arises from $\int_0^{2\pi} d\varphi\, e^{im\varphi} e^{-im'\varphi} = 2\pi\delta_{mm'}$

(19)
1/99

The normalization integral:

$$\int_{-1}^{1} d\mu \; P_\ell^{m} \; P_\ell^{m} =$$

$$= \frac{(-1)^m}{[(2)^\ell \ell!]^2} \frac{(\ell+m)!}{(\ell-m)!} \int_{-1}^{1} d\mu \left[\frac{d^{\ell+m}(\mu^2-1)^\ell}{d\mu^{\ell+m}}\right]\left[\frac{d^{\ell-m}(\mu^2-1)^\ell}{d\mu^{\ell-m}}\right]$$

If we integrate by parts m times we can move $d^{\ell+m}$ $d^{\ell-m}$ to d^ℓ d^ℓ. The boundary terms vanish at each step. A factor $(-1)^m$ comes from $\int uv = uv\big| - \int v\,du$

$$\int_{-1}^{1} d\mu \left(P_\ell^m\right)^2 = \frac{(\ell+m)!}{(\ell-m)!} \int_{-1}^{1} d\mu \underbrace{\left[\frac{1}{2^\ell \ell!}\frac{d^\ell}{d\mu^\ell}(\mu^2-1)^\ell\right]^2}_{[P_\ell(\mu)]^2}$$

Rodrigues

$$\int_{-1}^{1} d\mu \left(P_\ell^m\right)^2 = \frac{(\ell+m)!}{(\ell-m)!}\left(\frac{2}{2\ell+1}\right)$$

This is our normalization for the associated Legendre functions.

Now we can put the θ, ϕ angular parts together.

Fig.

$$P_\ell^m(\mu) = (1-\mu^2)^{m/2} \frac{d^m P_\ell(\mu)}{d\mu^m}$$

$\frac{d^m P_\ell(\mu)}{d\mu^m}$ ← polynomial of order ℓ

For $m > \ell$, this will vanish

$|m| \leq \ell$

$-\ell \leq m \leq \ell$

$P_{\ell m}$

	$m=0$	$m=1$	$m=2$	$m=3$
$\ell=0$	1	—	—	—
$\ell=1$	μ	$+\sqrt{1-\mu^2}$	—	—
$\ell=2$	$\frac{1}{2}(3\mu^2-1)$	$+3\mu\sqrt{1-\mu^2}$	$3(1-\mu^2)$	—
$\ell=3$	$\frac{\mu}{2}(5\mu^2-3)$	$+\frac{3}{2}(5\mu^2-1)\sqrt{1-\mu^2}$	$15\mu(1-\mu^2)$	$+15(1-\mu^2)^{3/2}$

$P_\ell^0 = P_\ell$

Always a factor of $\sqrt{1-\mu^2}$ $\Rightarrow$ $P_\ell^m(\pm 1) = 0$ for $m \neq 0$

- Phases agree Butkov, Weber + Arfken

Note: Some references define P_ℓ^m with factor $(-1)^m$ (eg Lea) (Figure)

Summarizing,

In spherical coordinates, the angular dependence of the Helmholtz equation is

$$\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right]=-\lambda Y$$

We wrote $Y=P(\theta)\,\Phi(\varphi)$ such that

$$\frac{d^2\Phi}{d\varphi^2}=-m^2\Phi \quad \text{and} \quad \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)-\frac{m^2}{\sin^2\theta}P+\lambda P=0$$

We imposed the conditions that:

Y be single valued $\longrightarrow$ m integer

Y be finite $\longrightarrow$ $\lambda=l(l+1)$; $l=0,1,2,\cdots$

The solutions are $\Phi(\varphi)\sim e^{im\varphi}$

$$P(\theta)\sim P_l^m(\theta)$$

such that $\int_0^{2\pi}d\varphi\,\frac{e^{-im'\varphi}}{\sqrt{2\pi}}\,\frac{e^{im\varphi}}{\sqrt{2\pi}}=\delta_{mm'}$

$$\int_{\cos\theta=-1}^{\cos\theta=+1}d\cos\theta\;P_l^m(\cos\theta)\,P_{l'}^m(\cos\theta)=\frac{2}{(2l+1)}\frac{(l+m)!}{(l-m)!}\delta_{ll'}$$

We define functions that are normalized to 1: (see below)

Spherical Harmonics

$$Y_l^m = (-1)^m \sqrt{\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, \frac{e^{im\varphi}}{\sqrt{2\pi}} \qquad m \geq 0$$

$$Y_l^m = \sqrt{\frac{2l+1}{2}\frac{(l-|m|)!}{(l+|m|)!}}\, P_l^{|m|}(\cos\theta)\, \frac{e^{im\varphi}}{\sqrt{2\pi}} \qquad m < 0$$

This is the Condon-Shortley phase convention.

(also used in Weber + Arfken ; Lea)

(But Kov uses Darwin phase : $Y_{lm}^{\text{Con-short}} = (-1)^m Y_{lm}^{\text{Dar}}$ $m \geq 0$; $Y_{lm}^{\text{Con-short}} = Y_{lm}^{\text{Dar}}$ $m < 0$)

We note that $Y_l^{m\,*}(\theta,\varphi) = (-1)^m Y_l^{-m}(\theta,\varphi)$.

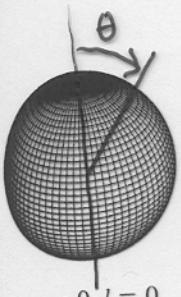
Thus the normalization of the spherical harmonics is given as:

$$\int_{\cos\theta=-1}^{\cos\theta=+1} d(\cos\theta) \int_0^{2\pi} d\varphi\; Y_l^{m\,*}(\theta,\varphi)\, Y_{l'}^{m'}(\theta,\varphi) = \delta_{ll'}\,\delta_{mm'}$$

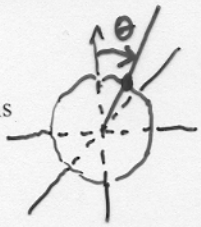
$$Y_l^0 = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\theta) \qquad \text{independent of } \varphi.$$

The defining equation for each figure is

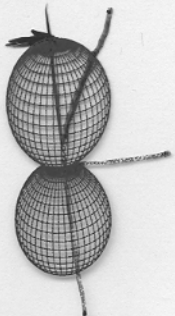
$$\{\mathrm{Re}[Y_l^m(\theta, \phi)]\}^2$$



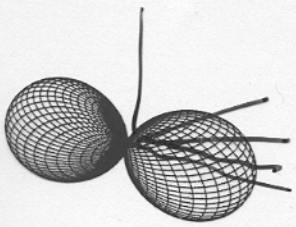
$m = 0, l = 0$



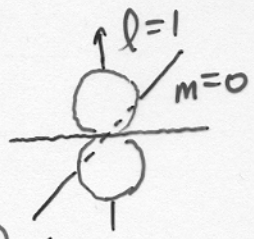
$l = m = 0$

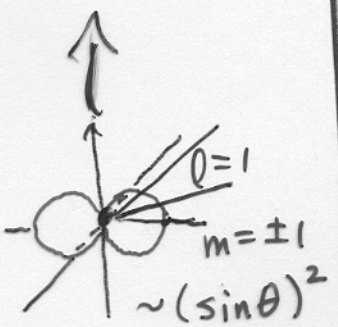


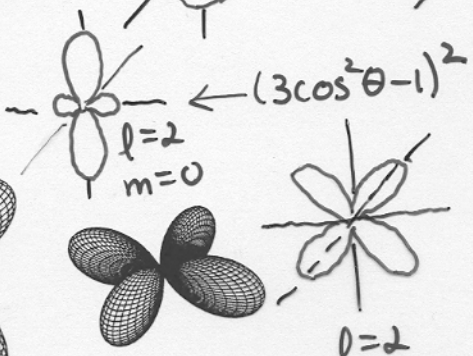
$m = 0, l = 1$



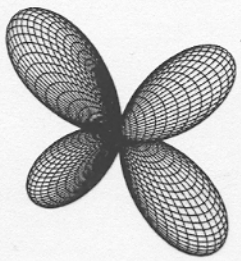
$m = 1, l = 1$





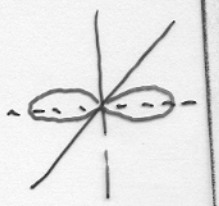


$m = 0, l = 2$

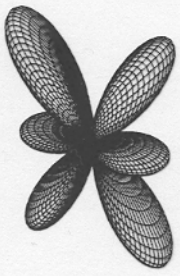


$m = 1, l = 2$

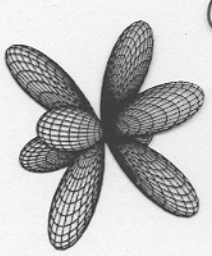
$m = 2, l = 2$



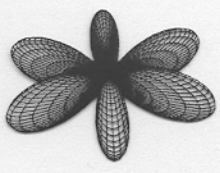
$m = 0, l = 3$



$m = 1, l = 3$



$m = 2, l = 3$

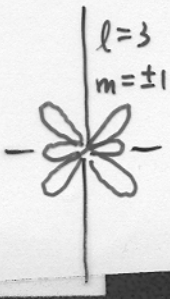


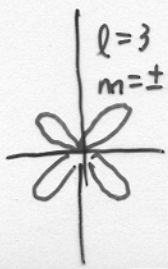
$m = 3, l = 3$

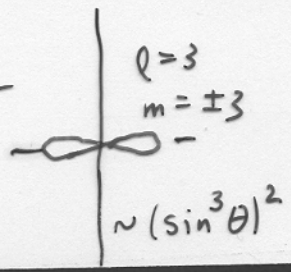
Lea

FIGURE 8.8. The first few spherical harmonics, with $l = 0, 1, 2, 3$; $m = 0, 1, 2, 3$. The real part is plotted here.

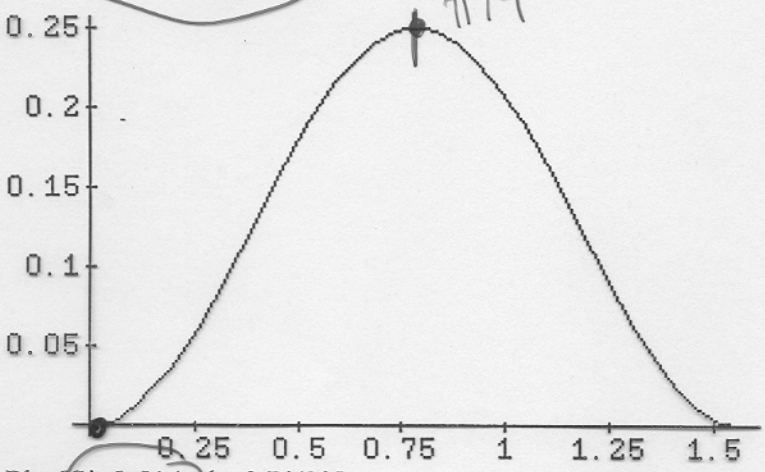
$l=3$ $m=0$

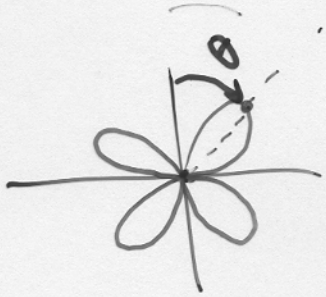






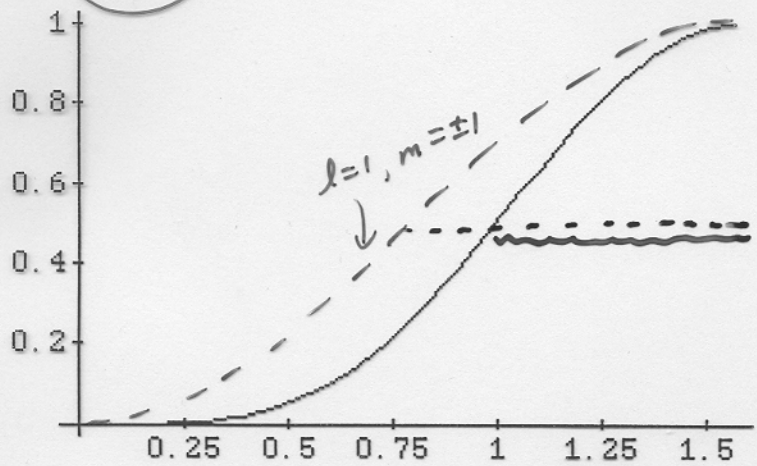
Plot[Cos[x]^2 Sin[x]^2, {x,0,Pi/2}]

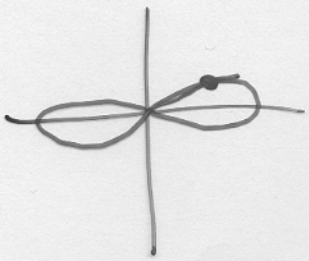




$l = 2 \quad m = \pm 1$

Plot[Sin[x]^4, {x,0,Pi/2}]

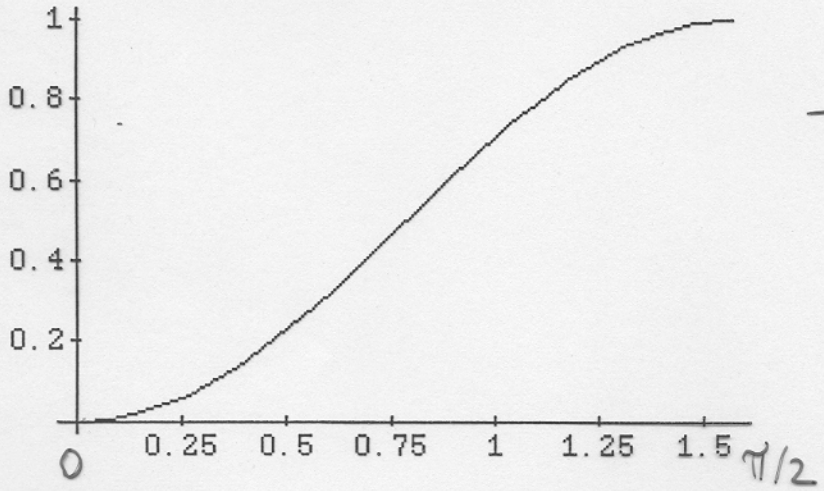


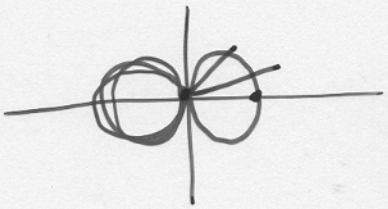


$l = 2 \quad m = \pm 2$

For higher quantum numbers, the width of this peaking goes like $l^{-1/2}$. Hence, approach classical picture of planar orbits.

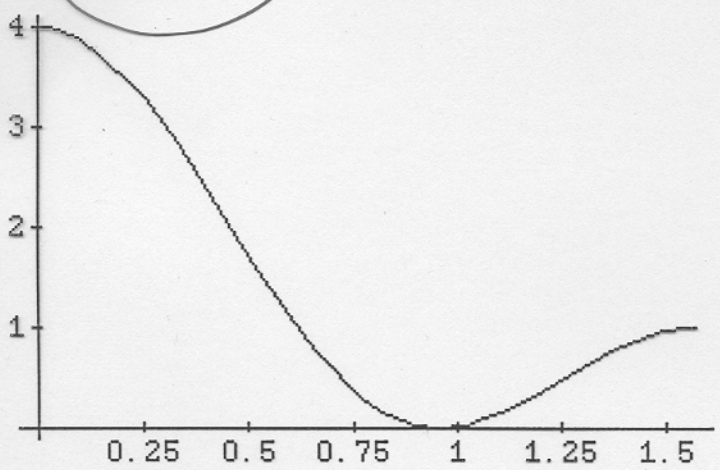
Plot[Sin[x]^2, {x,0,Pi/2}]

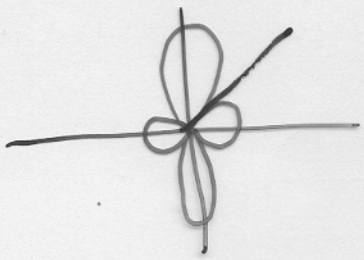




$\ell = 1 \quad m = \pm 1$

Plot[(3 Cos[x]^2 - 1)^2,{x,0,Pi/2}]





$\ell = 2 \quad m = 0$

Spherical Harmonics

$$Y_0^0(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$$

$$Y_1^1(\theta,\varphi) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi} \sim (x+iy)/r$$

$$Y_1^0(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta \sim z/r$$

$$Y_1^{-1}(\theta,\varphi) = +\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\varphi} \sim (x-iy)/r$$

$$Y_2^2(\theta,\varphi) = \sqrt{\frac{5}{96\pi}}\,3\sin^2\theta e^{2i\varphi} \sim (x+iy)^2/r^2$$

$$Y_2^1(\theta,\varphi) = -\sqrt{\frac{5}{24\pi}}\,3\sin\theta\cos\theta e^{i\varphi} \sim z(x+iy)/r^2$$

$$Y_2^0(\theta,\varphi) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \sim (2z^2-x^2-y^2)/r^2$$

$$Y_2^{-1}(\theta,\varphi) = +\sqrt{\frac{5}{24\pi}}\,3\sin\theta\cos\theta e^{-i\varphi} \sim z(x-iy)/r$$

$$Y_2^{-2}(\theta,\varphi) = \sqrt{\frac{5}{96\pi}}\,3\sin^2\theta e^{2i\varphi} \sim (x-iy)^2/r^2$$

$$Y_3^3(\theta,\varphi) = -\frac{1}{8}\sqrt{\frac{35}{\pi}}\sin^3\theta e^{3i\varphi} \sim (x+iy)^3/r^3$$

$$Y_3^2(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\theta\cos\theta e^{2i\varphi} \sim z(x+iy)^2/r^3$$

$$Y_3^1(\theta,\varphi) = -\frac{1}{8}\sqrt{\frac{21}{\pi}}(5\cos^2\theta-1)\sin\theta e^{i\varphi} \sim (4z^2-x^2-y^2)\frac{(x+iy)}{r^3}$$

$$Y_3^0(\theta,\varphi) = \sqrt{\frac{7}{4\pi}}\cos\theta\left(\frac{5}{2}\cos^2\theta - \frac{3}{2}\right) \sim (2z^3-3zx^2-3zy^2)/r^3$$

$$\vdots$$

The spherical harmonics form a complete set of functions in the angular coordinates, θ and φ, on a sphere.

We can expand a function $f(\theta,\varphi)$ as:

$$f(\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=+l} a_{lm} Y_l^m(\theta,\varphi)$$

$$4\pi = d\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta$$

The coefficients are determined using the orthonormality property:

$$a_{lm} = \int_{\cos\theta'=-1}^{\cos\theta'=+1} d(\cos\theta') \int_0^{2\pi} d\varphi' \; Y_l^{m*}(\theta',\varphi') f(\theta',\varphi') \quad **$$

$$\int_{-1}^{+1} d\cos\theta' = \int_0^{\pi} d\theta \sin\theta$$

Putting ** back into *:

$$f(\theta,\varphi) = \int_0^{4\pi} d\theta' \int_0^{2\pi} d\varphi' \left[\sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \sin\theta \, Y_l^m(\theta,\varphi) Y_l^{m*}(\theta',\varphi') \right] f(\theta',\varphi')$$

So we identify the completeness relation:

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{+l} Y_l^m(\theta,\varphi) Y_l^{m*}(\theta',\varphi') = \frac{\delta(\theta-\theta')\,\delta(\varphi-\varphi')}{\sin\theta}$$

Addition Theorem for Spherical Harmonics

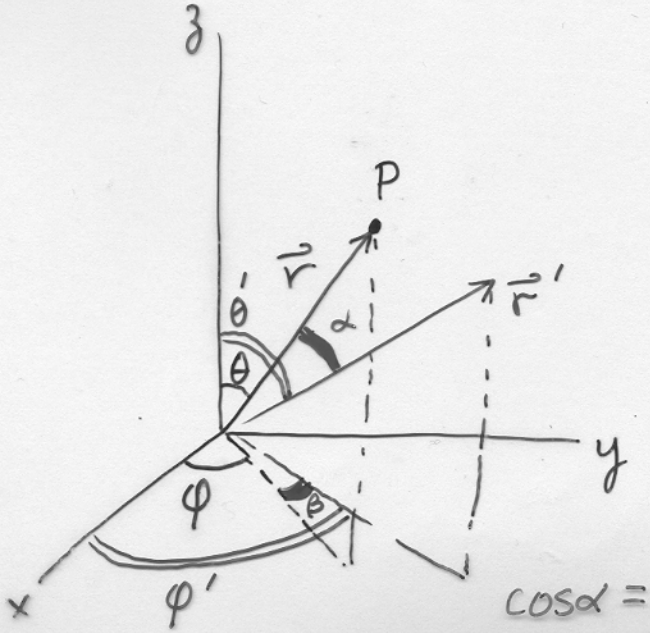
This theorem gives us a relation between spherical harmonics referred to two differently oriented coordinate systems. It relies on the invariance of the Laplacian operator, under rotations of the coord system.

$$\nabla^2 \psi(x,y,z) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \psi(x,y,z)$$

$$\nabla'^2 \psi(x',y',z') = \underbrace{\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right)}_{\text{scalar operator}} \psi(x',y',z') \qquad \nabla^2 = \vec{\nabla}\cdot\vec{\nabla}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\,) + \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi^2}\right]$$

r is a scalar so the angular part of the ∇^2 operator is invariant.



$$L = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi^2}\right]$$

$$= L' = \frac{1}{\sin\theta'}\left[\frac{\partial}{\partial\theta'}\left(\sin\theta'\frac{\partial}{\partial\theta'}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi'^2}\right]$$

Eigenfns $L Y_{lm}(\theta,\varphi) = -l(l+1) Y_{lm}(\theta,\varphi)$

$L' Y_{lm}(\theta',\varphi') = -l(l+1) Y_{lm}(\theta',\varphi')$

$$\cos\alpha = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$$

Thus a spherical harmonic of degree l, Y_{lm}, expressed in the coords (x, y, z) [angles (θ, ϕ)] is also a spherical harmonic of the same degree expressed in the coords $(x'; y'; z')$ [angles (θ', ϕ')].

It doesn't matter how we choose to orient our coordinate system.

So we must be able to express spherical harmonics of same degree $Y_{lm}(\theta', \varphi')$ as linear combinations of $Y_{lm'}(\theta, \varphi)$.

$$P_l(\mu) = P_l(\cos\theta)$$

Consider the particular function $P_l(\cos\alpha) = P_l(\alpha)$. This is a spherical harmonic of degree l. We must be able to express this fn as a linear combination of $Y_{lm}(\theta, \varphi)$.

$$P_l(\alpha) = \sum_{m=-l}^{+l} a_m Y_{lm}(\theta, \varphi)$$

Alternately, we expect to be able to write

$$P_l(\alpha) = \sum_{m'=-l}^{+l} a'_{m'} Y_{lm'}(\theta', \varphi')$$

So ultimately we expect something like

$$P_l(\alpha) = \sum_{m,m'} a_{mm'} Y_{lm}(\theta, \varphi)\, Y_{lm'}(\theta', \varphi')$$

Start with $$P_\ell(\alpha) = \sum_{m=-\ell}^{+\ell} a_m Y_{\ell m}(\theta,\varphi)$$

If we proceed in the usual way

(†) $$a_m = \int d\Omega\, Y_{\ell m}^*(\theta,\varphi)\, P_\ell(\alpha)$$

$d\Omega = d\cos\theta\, d\varphi$
solid angle

* invariant under rotn.

Let's try to eraluate this expression by writing $Y_{\ell m}^*(\theta,\varphi)$ as an expansion in $Y_{\ell m'}(\alpha,\beta)$. (That is, in terms of a coord system with z-axis along $\vec{r}$.)

(#) $$Y_{\ell m}^*(\theta,\varphi) = \sum_{m'=-\ell}^{+\ell} b_{m'} Y_{\ell m'}(\alpha,\beta)$$

If we again proceed as usual

$$b_{m'} = \int d\Omega_\alpha\, Y_{\ell m'}^*(\alpha,\beta)\, Y_{\ell m}^*(\theta,\varphi)$$

$d\Omega_\alpha = d\cos\alpha\, d\beta = d\Omega$

In particular look at b_0:

$$b_0 = \int d\Omega_\alpha\, Y_{\ell 0}^*(\alpha,\beta)\, Y_{\ell m}^*(\theta,\varphi)$$

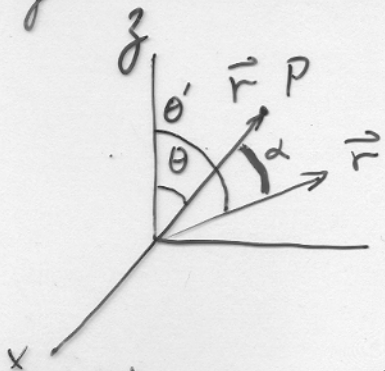
But $$Y_{\ell 0}^*(\alpha,\beta) = Y_{\ell 0}(\alpha,\beta) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\alpha)$$

$$b_0 = \int d\Omega_\alpha \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\alpha)\, Y_{\ell m}^*(\theta,\varphi) = \int d\Omega \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\alpha)\, Y_{\ell m}^*(\theta,\varphi)$$

(†) But this is just $\sqrt{\frac{2\ell+1}{4\pi}}\, a_m$.

So $a_m = \sqrt{\frac{4\pi}{2l+1}}\, b_0$

From (#): $Y_{lm}^*(\theta,\varphi) = \sum_{m'=-l}^{+l} b_{m'}\, Y_{lm'}(\alpha,\beta)$, we can get an alternate expression for b_0.



If $\vec{r}$ and $\vec{r}'$ coincide, then $\alpha = 0$ and $(\theta,\varphi) = (\theta',\varphi')$.

$Y_{lm'}(0,\beta)$ is only nonzero for $m' = 0$.

since

$Y_{lm}(\theta,\varphi)$ always has at least one factor of $\sin\theta$ for $m \neq 0$.

Thus (#) simplifies to

specifically change to (θ',φ') to indicate $\vec{r} = \vec{r}'$ here.

$$Y_{lm}^*(\theta,\varphi) = Y_{lm}^*(\theta',\varphi') = b_0\, Y_{l0}(0,\beta)$$

$$Y_{l0}(0,\beta) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\alpha=0) = \sqrt{\frac{2l+1}{4\pi}}$$

$$P_l(\cos\alpha = 0) = P_l(\mu = 1) = 1$$

Thus $Y_{lm}^*(\theta',\varphi') = \sqrt{\frac{2l+1}{4\pi}}\, b_0$

So $a_m = \left(\frac{4\pi}{2l+1}\right) Y_{lm}^*(\theta',\varphi')$

So finally

$$P_\ell(\alpha) = \sum_{m=-\ell}^{+\ell} a_m Y_{\ell m}(\theta,\varphi)$$

$$= \sum_{m=-\ell}^{+\ell} \left(\frac{4\pi}{2\ell+1}\right) Y_{\ell m}^*(\theta',\varphi') Y_{\ell m}(\theta,\varphi)$$

This is the addition theorem.

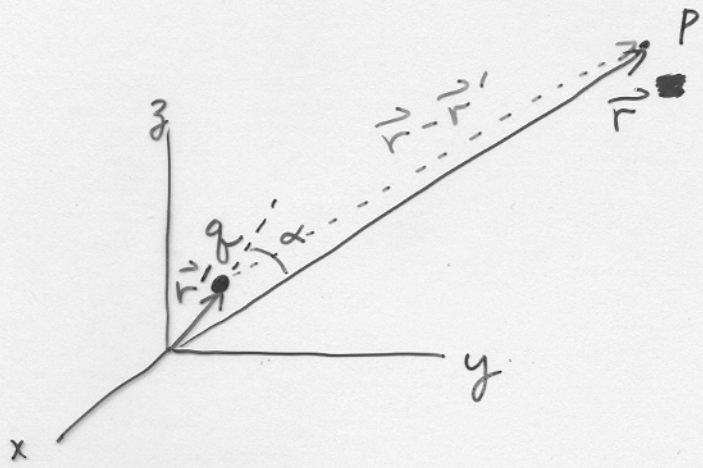
If $\hat{r}$ and $\hat{r}'$ are unit vectors in the $\vec{r}$ and $\vec{r}'$ directions, respectively, then $\cos\alpha = \hat{r}\cdot\hat{r}'$

$$P_\ell(\hat{r}\cdot\hat{r}') = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{+\ell} Y_{\ell m}^*(\hat{r}') \; Y_{\ell m}(\hat{r})$$

For the special case $\alpha = 0$, $P_\ell(0) = 1$ and $(\theta,\varphi) = (\theta',\varphi')$

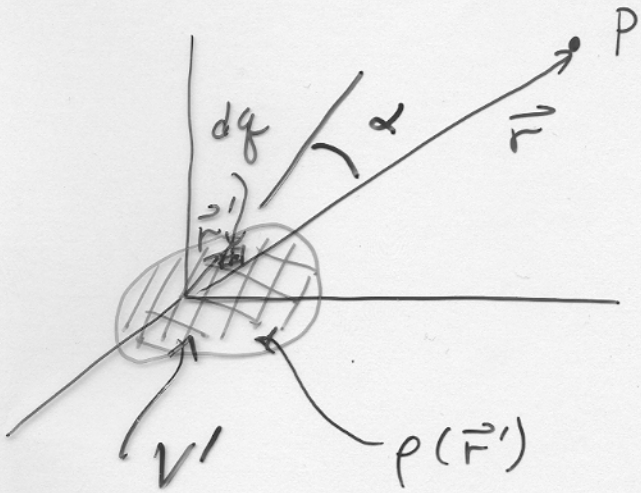
so $$\sum_{m=-\ell}^{+\ell} |Y_{\ell m}(\theta,\varphi)|^2 = \frac{2\ell+1}{4\pi}$$

We can now use our addition theorem in combination with the generating function of the Legendre polynomials to do a multipole expansion. In a multipole expansion, a potential $\varphi(\vec{r})$ is expressed as a sum of terms, each of which has a distinct angular and radial dependence. It is applicable for electromagnetic and gravitational systems. We will use em example but can equally get mass multipoles.



For a point charge q located at position $\vec{r}'$, the potential at the point P at location $\vec{r}$ is

$$\varphi(\vec{r}) = \frac{q}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|}$$



For a charge distribution of density $\rho(\vec{r}')$, one must integrate over the distribution to calculate the contribution to the potential at P $(\vec{r})$.

$$\varphi(\vec{r}) = \int_{V'} d^3x' \frac{\rho(\vec{r}')}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|}$$

This assumes that the charge density is localized within some region of radius $< r$. So $r' < r$ throughout V'!

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{[r^2 - 2rr'\cos\alpha + r'^2]^{1/2}} = \frac{1}{r\left[1 - 2\left(\frac{r'}{r}\right)\cos\alpha + \left(\frac{r'}{r}\right)^2\right]^{1/2}}$$

This is of generating fn form:

$$[1 - 2t\mu + t^2]^{-1/2} = \sum_{l=0}^{\infty} P_l(\mu = \cos\alpha)\, t^l$$

$t = \left(\frac{r'}{r}\right)$

So $$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\alpha)$$

$r' < r$, $r_<$, $r_>$

$$P_l(\cos\alpha) = \frac{4\pi}{2l+1}\sum_{m=-l}^{+l} Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi)$$

Y_{lm}: angles of $\vec{r}$

Y^*_{lm}: angular coords of $\vec{r}'$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{4\pi}{(2l+1)}\frac{(r')^l}{r^{l+1}}\, Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi)$$

$$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{4\pi}{(2l+1)}\left[\int_{V'} d^3x'\,(r')^l\, Y^*_{lm}(\theta',\varphi')\,\rho(\vec{r}')\right]\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}$$

$\equiv q_l^m$

The multipole moments characterize the charge distribution.

distinct radial and angular dependence for each term in the series

$l = m = 0$: $Y^*_{00} = 1/\sqrt{4\pi}$

$$q_0^0 = \int_{V'} d^3x'\,\frac{\rho(\vec{r}')}{\sqrt{4\pi}} = \frac{q}{\sqrt{4\pi}} \leftarrow \text{total charge in } V'$$

First term in φ: $\frac{1}{4\pi\epsilon_0}\,\frac{4\pi}{1}\left(\frac{q}{\sqrt{4\pi}}\right)\frac{1}{\sqrt{4\pi}\,r} = \frac{q}{4\pi\epsilon_0}$ monopole

Recall:

$$x' = r'\sin\theta'\cos\varphi'$$
$$y' = r'\sin\theta'\sin\varphi'$$
$$z' = r'\cos\theta'$$

$\ell = 1$:

$m=1$: $q_1^{1} = \int d^3x' \underbrace{Y_{11}^{*}(\theta',\varphi')}\, r'\rho(\vec{r}')$

$$-\sqrt{\frac{3}{8\pi}}\sin\theta' e^{-i\varphi'}$$

$$= -\sqrt{\frac{3}{8\pi}}\int d^3x' \; r'\left[\sin\theta'(\cos\varphi' - i\sin\varphi')\right]\rho(\vec{r}')$$

$$= -\sqrt{\frac{3}{8\pi}}\int d^3x' \; (x' - iy')\rho(\vec{r}')$$

For $\vec{p} = \int d^3x' \, \vec{r}'\rho(\vec{r}')$

$$q_1^{1} = -\sqrt{\frac{3}{8\pi}}\,(p_x - i p_y)$$

$m=0$: $q_1^{0} = \int d^3x' \underbrace{Y_{10}^{*}(\theta',\varphi')}\, r'\rho(\vec{r}')$

$$\sqrt{\frac{3}{4\pi}}\underbrace{\cos\theta'}_{z'}$$

$$= \sqrt{\frac{3}{4\pi}}\int d^3x \; z'\rho(\vec{r}') = \sqrt{\frac{3}{4\pi}}\, p_z$$

We recognize $\vec{p}$ as the dipole moment for a charge distribution given by $\rho(\vec{r}')$.

$\ell = 2$: $m = 0$ Case

$$q_2^0 = \int d^3x' \, Y_2^{0*} r'^2 \rho(\vec{r}') = \int d^3x' \sqrt{\frac{5}{4\pi}} \frac{1}{2} \underbrace{(3\cos^2\theta' - 1) r'^2}_{3z'^2 - r'^2} \rho(\vec{r}')$$

Defining $Q_{ij} = \int d^3x' \, (3x_i' x_j' - r'^2 \delta_{ij}) \rho(\vec{r}')$ quadrupole moment tensor

$$q_2^0 = \frac{1}{2}\sqrt{\frac{5}{4\pi}} Q_{33}$$

$$q_2^1 = \int d^3x' \, Y_2^{1*} r'^2 \rho(\vec{r}') = \int d^3x' \left(-\sqrt{\frac{5}{24\pi}}\right) \underbrace{3\sin\theta'\cos\theta' e^{-i\varphi'} r'^2}_{3z'(x' - iy')} \rho(\vec{r}')$$

$$= -\sqrt{\frac{5}{24\pi}} (Q_{13} - i Q_{23})$$

$$q_2^2 = \int d^3x' \, Y_2^{2*} r'^2 \rho(\vec{r}') = \int d^3x' \sqrt{\frac{5}{96\pi}} \underbrace{3\sin^2\theta' e^{-2i\varphi'} r'^2}_{\frac{[r'\sin\theta'(\cos\varphi' - i\sin\varphi')]^2}{(x' - iy')^2}} \rho(\vec{r}')$$

$$= \sqrt{\frac{5}{96\pi}} (Q_{11} - 2iQ_{12} - Q_{22})$$

So Q_{ij} is recognized as the quadrupole moment for the charge distribution.

Note $Q_{ij} = Q_{ji}$ and $\sum_{i=1}^{3} Q_{ii} = 0$ so we have

$3 \times 3 - 3 - 1 = 5$ independent real numbers to specify quadrupole moment.

Put these results for the q_ℓ^m and put them back into

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \frac{4\pi}{2\ell+1} q_\ell^m \frac{Y_{\ell m}(\theta,\varphi)}{r^{\ell+1}}$$

Note that since $q_\ell^m \sim Y_{\ell m}^*$, $q_\ell^{-m} = (-1)^m q_\ell^{m*}$.

After expressing the $Y_{\ell m}(\theta,\varphi)$ as on pg 177 and some algebra:

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \left\{ \frac{q}{r} + \frac{\vec{p}\cdot\vec{r}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij} \frac{x_i x_j}{r^5} + \cdots \right\}$$

monopole — dipole — quadrupole

This is a convenient and standard way to characterize charge distributions, including for atomic and nuclear systems. Also used to characterize gravitational systems including, for instance, planetary systems.

From φ above, can get electric field corresponding to the charge distribution as:

$$\vec{E}(\vec{r}) = -\vec{\nabla}\varphi(\vec{r})$$

After this investigation of the angular part of the solution to Helmholtz eqn $\nabla^2 u + k^2 u = 0$ in spherical coordinates, we can return to incorporate the radial part.

$$u = R(r)\, Y(\theta, \varphi)$$

$Y(\theta, \varphi) = Y_{lm}(\theta, \varphi)$ the spherical harmonics

$$\frac{1}{r}\frac{d^2}{dr^2}(rR) + k^2 R - \frac{\lambda}{r^2} R = 0$$

But $\lambda = l(l+1)$ with $l = 0, 1, 2, \ldots$ in order to keep Y_{lm} finite.

$$\frac{1}{r}\frac{d^2}{dr^2}(rR) + k^2 R - \frac{l(l+1)}{r^2} R = 0$$

Consider first the simpler case of Laplace's equation $\nabla^2 u = 0 \longrightarrow k^2 = 0$.

$$\frac{1}{r}\frac{d^2}{dr^2}(rR) - \frac{l(l+1)}{r^2} R = 0$$

Back on pg (40-43) we noted that trying a power law solution r^α lead to the result that $\alpha = \frac{1}{2}\left(-1 \pm \sqrt{1 + 4\lambda}\right)$, with $\lambda = l(l+1)$.

Two roots are $\alpha = l$ and $\alpha = -(l+1)$

$$\boxed{R(r) = A r^l + B r^{-(l+1)}}$$

So for $\nabla^2 u_{lm} = 0$ the form of the solution is

$$u_{lm} = \left[A r^l + \frac{B}{r^{l+1}} \right] Y_{lm}(\theta, \varphi) .$$

A general solution for $\nabla^2 \psi = 0$ is a superposition

$$\psi(\vec{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \left[A_{lm} r^l + \frac{B_{lm}}{r^{l+1}} \right] Y_{lm}(\theta, \varphi) .$$

A_{lm} and B_{lm} determined by boundary conditions.

Consider the set of general examples in Wyld Sec 3.9.

Case 1 $r \le a$ $\longrightarrow$ B_{lm} vanish
Case 2 $r \ge a$ $\longrightarrow$ A_{lm} vanish
} ψ finite ($r \to 0$, $r \to \infty$)
Case 3 $a \le r \le b$ $\longrightarrow$ generally need both terms

For Case 1 $\boxed{r \le a}$ and Case 2 $\boxed{r \ge a}$:

Consider two types of boundary condition

i) ψ is specified at $r = a$

ii) $\frac{\partial \psi}{\partial r}$ is specified at $r = a$

$$\int d\Omega\, Y^*_{l'm'}(\theta,\varphi)\, Y_{lm}(\theta,\varphi) = \delta_{ll'}\delta_{mm'} \quad (192)$$

Case (1i) $\quad r \le a \quad \psi(a,\theta,\varphi) = u(\theta,\varphi)$ ← given

$$\boxed{\psi(r,\theta,\varphi) = \sum_{l,m} A_{lm}\, r^l\, Y_{lm}(\theta,\varphi)}$$

$$\psi(r=a,\theta,\varphi) = u(\theta,\varphi) = \sum_{l,m} A_{lm}\, a^l\, Y_{lm}(\theta,\varphi)$$

$$\int d\Omega\, Y^*_{l'm'}(\theta,\varphi)$$

$$A_{lm} = \frac{1}{a^l}\int d\Omega\, Y^*_{lm}(\theta,\varphi)\, u(\theta,\varphi)$$

Case (1ii) $\quad r \le a \quad \left.\frac{\partial\psi}{\partial r}\right|_{r=a} = v(\theta,\varphi)$ ← given

$$\frac{\partial\psi}{\partial r}(r,\theta,\varphi) = \sum_{l,m} l\, A_{lm}\, r^{l-1}\, Y_{lm}(\theta,\varphi)$$

$$\left.\frac{\partial\psi}{\partial r}\right|_{r=a} = v(\theta,\varphi) = \sum_{l,m} l\, A_{lm}\, a^{l-1}\, Y_{lm}(\theta,\varphi)$$

$$\int d\Omega\, Y^*_{l'm'}(\theta,\varphi)$$

$$A_{lm} = \frac{1}{l\, a^{l-1}}\int d\Omega\, Y^*_{lm}(\theta,\varphi)\, v(\theta,\varphi)$$

Note, this holds for $l \neq 0$. So A_{00} is not specified with this boundary condition.

Case (2i) $\quad r \geq a \quad \quad \psi(a,\theta,\varphi) = u(\theta,\varphi)$

$$\boxed{\psi(r,\theta,\varphi) = \sum_{l,m} B_{lm} \frac{1}{r^{l+1}} Y_{lm}(\theta,\varphi)}$$

$$\psi(r=a,\theta,\varphi) = u(\theta,\varphi) = \sum_{l,m} B_{lm} \frac{1}{a^{l+1}} Y_{lm}(\theta,\varphi)$$

$$B_{lm} = a^{l+1} \int d\Omega \; Y_{lm}^{*}(\theta,\varphi)\, u(\theta,\varphi)$$

Case (2ii) $\quad r \geq a \quad \quad \left.\frac{\partial \psi}{\partial r}\right|_{r=a} = v(\theta,\varphi)$

$$\frac{\partial \psi}{\partial r}(r,\theta,\varphi) = \sum_{l,m} B_{lm} \left(-\frac{(l+1)}{r^{l+2}} \right) Y_{lm}(\theta,\varphi)$$

$$\left.\frac{\partial \psi}{\partial r}\right|_{r=a} = v(\theta,\varphi) = -\sum_{l,m} B_{lm} \frac{(l+1)}{a^{l+2}} Y_{lm}(\theta,\varphi)$$

$$B_{lm} = -\frac{a^{l+2}}{(l+1)} \int d\Omega \; Y_{lm}^{*}(\theta,\varphi)\, v(\theta,\varphi)$$

Case 3 $a \le r \le b$

$$\psi(r,\theta,\varphi) = \sum_{l,m} \left[A_{lm} r^l + B_{lm} \frac{1}{r^{l+1}} \right] Y_{lm}(\theta,\varphi)$$

Specify ψ on the boundaries $r=a$ and $r=b$.

$$\psi(a,\theta,\varphi) = f(\theta,\varphi)$$

$$\psi(b,\theta,\varphi) = g(\theta,\varphi)$$

$$\psi(a,\theta,\varphi) = f(\theta,\varphi) = \sum_{l,m} \left[A_{lm} a^l + B_{lm} \frac{1}{a^{l+1}} \right] Y_{lm}(\theta,\varphi)$$

(1) $$A_{lm} a^l + B_{lm} \frac{1}{a^{l+1}} = \int d\Omega \; Y_{lm}^*(\theta,\varphi) f(\theta,\varphi)$$

$$\psi(b,\theta,\varphi) = g(\theta,\varphi) = \sum_{l,m} \left[A_{lm} b^l + B_{lm} \frac{1}{b^{l+1}} \right] Y_{lm}(\theta,\varphi)$$

(2) $$A_{lm} b^l + B_{lm} \frac{1}{b^{l+1}} = \int d\Omega \; Y_{lm}^*(\theta,\varphi) g(\theta,\varphi)$$

(1) & (2) For each value of l, m we have two eqns in two unknowns A_{lm} and B_{lm}. Solve.

Other b.c.'s: $\psi(a)$ and $\left.\frac{\partial \psi}{\partial r}\right|_b$

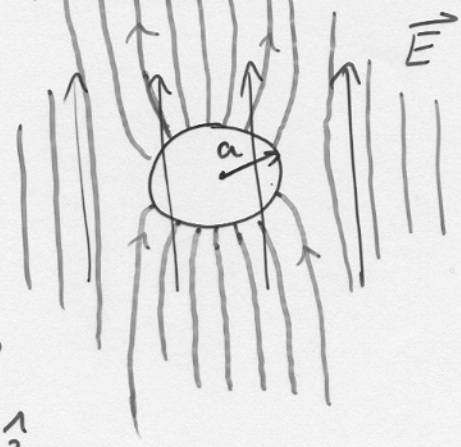
$\left.\frac{\partial \psi}{\partial r}\right|_a$ and $\psi(b)$

$\left.\frac{\partial \psi}{\partial r}\right|_a$ and $\left.\frac{\partial \psi}{\partial r}\right|_b$

Two classic examples given in Wyld.

Sec 3.10 Conducting sphere in a uniform external electric field. Find $\phi(r,\theta,\phi)$

$\vec{E} = E_0 \hat{z}$



$\nabla^2 \phi = 0$

Far from the sphere, the field remains uniform. $\xrightarrow{r \to \infty}$

$$\vec{E} = E_0 \hat{z} = -\vec{\nabla}\phi = -\frac{\partial \phi}{\partial z}\hat{z}$$

$$\frac{\partial \phi}{\partial z} = -E_0 \Rightarrow \phi = -E_0 z + \text{constant}$$

$$\phi \xrightarrow[r\to\infty]{} -E_0 r\cos\theta + \underbrace{\text{constant}}$$

We will drop this.

So one b.c. is $\phi \xrightarrow[r\to\infty]{} -E_0 r\cos\theta + \text{const.}$

$*$

↑ Note: Not same as $\phi \to 0$ as $r \to \infty$.

The sphere is conducting so, in equilibrium, the potential must be constant on its surface.

$$\phi(r=a,\theta,\phi) = V = \frac{Q}{4\pi\epsilon_0 a}$$

↑

This is second b.c.

In general we know

$$\varphi(r,\theta,\varphi) = \sum_{l,m}\left[A_{lm}\, r^l + \frac{B_{lm}}{r^{l+1}}\right] Y_{lm}$$

For the $r\to\infty$ condition, we can ignore the terms in $\frac{1}{r^{l+1}}$. (Will get B_{lm} from other b.c.)

$$\sum_{l,m} A_{lm} r^l Y_{lm} \xrightarrow[r\to\infty]{} -E_0 r \underbrace{\cos\theta}_{\cos\theta = \sqrt{\frac{4\pi}{3}}\, Y_{1,0}} + \underbrace{\text{constant}}_{\sim Y_{00} = \frac{1}{\sqrt{4\pi}}}$$

$$\int d\Omega\, Y^*_{l'm'}$$

$$A_{l'm'} r^{l'} = -E_0 r \sqrt{\frac{4\pi}{3}}\, \delta_{l',1}\, \delta_{m'0} + C\, \delta_{l'0}\, \delta_{m'0}$$

So only A_{10} and A_{00} can be nonvanishing.

↑ Drop A_{00} now since it's just a constant.

$$\boxed{A_{10} = -E_0\sqrt{\frac{4\pi}{3}}}$$

$$\varphi(r,\theta,\varphi) = \underbrace{A_{10}}_{-E_0\sqrt{\frac{4\pi}{3}}} r\, Y_{10} + \sum_{l,m} \frac{B_{lm}}{r^{l+1}} Y_{lm}$$

Second b.c.

$$\varphi(a,\theta,\varphi) = V = -E_0\sqrt{\frac{4\pi}{3}}\, a\, Y_{10} + \sum_{l,m} \frac{B_{lm}}{a^{l+1}} Y_{lm}$$

$$\int d\Omega\, Y^*_{l'm'} V = \int d\Omega\, Y^*_{l'm'} \left[-E_0\sqrt{\frac{4\pi}{3}}\, a\, Y_{10} + \sum_{l,m} \frac{B_{lm}}{a^{l+1}} Y_{lm} \right]$$

First treat $l=1$, $m=0$ term

$$\int d\Omega \left(Y^*_{10}\right) V = -E_0\, a\sqrt{\frac{4\pi}{3}} + \frac{B_{10}}{a^2}$$

$$Y^*_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad \int_{-1}^{1} d\cos\theta\, \cos\theta = \frac{\cos^2\theta}{2}\Big|_{-1}^{1} = 0$$

LHS = 0 $\quad\therefore\quad \boxed{B_{10} = a^3 E_0\sqrt{\frac{4\pi}{3}}}$

Now treat the case $(l,m) \neq (1,0)$

[illegible]

$$\int d\Omega\, Y^*_{lm} V = \frac{B_{lm}}{a^{l+1}}$$

← Use properties of Y_{lm}

$$Y_{lm}^* \sim e^{-im\varphi}$$

$$\text{So} \quad \text{LHS} \sim \int_0^{2\pi} d\varphi \; e^{-im\varphi}$$

For $m=0$, this is nonvanishing but for $m \neq 0$, LHS $= 0$.

Thus only B_{00} contributes to $\int d\Omega \, V Y_{lm}^* = \frac{B_{lm}}{a^{l+1}}$

$$B_{00} = aV \int d\Omega \frac{1}{\sqrt{4\pi}} = \boxed{\sqrt{4\pi}\, aV = B_{00}}$$

Thus, in $\varphi(r,\theta,\varphi) = \sum_{l,m} \left[A_{lm} r^l + \frac{B_{lm}}{r^{l+1}} \right] Y_{lm}$

only $\begin{cases} A_{10} = -E_0 \sqrt{\frac{4\pi}{3}} \\ B_{00} = \sqrt{4\pi}\, aV \\ B_{10} = a^3 E_0 \sqrt{\frac{4\pi}{3}} \end{cases}$ contribute. Use $Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$\varphi(r,\theta,\varphi) = \frac{\sqrt{4\pi}\, aV}{r} \frac{1}{\sqrt{4\pi}} + \left[-E_0 \sqrt{\frac{4\pi}{3}} r + \frac{a^3 E_0}{r^2} \sqrt{\frac{4\pi}{3}} \right] \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$= \frac{Va}{r} + E_0 \left[-r + \frac{a^3}{r^2} \right] \cos\theta$$

$$= \frac{Q}{4\pi\epsilon_0 r} + E_0 \left[-r + \frac{a^3}{r^2} \right] \cos\theta$$

From the multipole perspective, we have

$$\varphi(r,\theta,\varphi) = \frac{Q}{4\pi\epsilon_0 r} + \frac{E_0 a^3 r\cos\theta}{r^3} - E_0 r\cos\theta$$

$\frac{Q}{4\pi\epsilon_0 r}$: contribution to the potential due to charged sphere

Monopole term

$\frac{E_0 a^3 r\cos\theta}{r^3}$

$-E_0 r\cos\theta$: contribution to potential due to external Electric field $\vec{E} = E_0\hat{z}$

$$\frac{E_0 a^3 z}{r^3} = \frac{(E_0 a^3 \hat{z})\cdot\vec{r}}{r^3} = \frac{\vec{p}\cdot\vec{r}}{r^3}$$

Dipole moment $\vec{p} = E_0 a^3 \hat{z}$ induced in the sphere ~~due to~~ by external electric field.

Sec 3.11 Wyld Flow of an Incompressible Fluid around a Spherical Object

The continuity equation $\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot(\rho\vec{v})$ is obeyed.

ρ is density of the fluid.

$\vec{v}$ is velocity of the fluid.

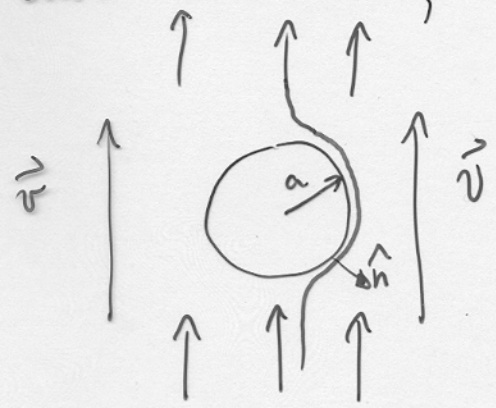
ρ is constant and uniform. So $\vec{\nabla}\cdot\vec{v} = 0$.

For irrotational flow $\vec{\nabla}\times\vec{v} = 0$. Thus $\vec{v}$ can be expressed as the gradient of a scalar:

$$\vec{v} = -\vec{\nabla}\psi$$

ψ is the "velocity potential"

Since $\vec{\nabla}\cdot\vec{v} = 0$, ψ obeys Laplace's eqn $\nabla^2\psi = 0$.



One b.c. is:
For $r \to \infty$, $\vec{v} = v_0\hat{z}$

$$v_0\hat{z} = -\vec{\nabla}\psi = -\frac{\partial\psi}{\partial z}\hat{z}$$

$$\psi = -v_0 z + \text{constant}$$

$$\psi \underset{r\to\infty}{\longrightarrow} -v_0 r\cos\theta + \text{constant}.$$

This is identical to our electrostatics problem so

$$\psi(r,\theta,\varphi) = A_{00}Y_{00} + A_{10}\,r\,Y_{10} + \sum_{l,m} B_{lm}\frac{1}{r^{l+1}}Y_{lm}$$

A_{00} is undetermined. Denote $A_{00} Y_{00} = c$ = constant

A_{10} is such that $A_{10} r Y_{10} = -v_0 r \cos\theta$

$$A_{10} = -\frac{v_0 \cos\theta}{Y_{10}}$$

$$\psi(r,\theta,\varphi) = c + \underbrace{-v_0 r\cos\theta}_{= A_{10} r Y_{10}} + \sum_{l,m} B_{lm} \frac{1}{r^{l+1}} Y_{lm}$$

The other b.c. is that the velocity is tangential to the surface of the sphere.

$$\hat{n}\cdot\vec{v}\Big|_{r=a} = 0 = -\hat{n}\cdot\vec{\nabla}\psi\Big|_{r=a} = -\hat{r}\cdot\vec{\nabla}\psi\Big|_{r=a}$$

Thus $$\frac{\partial\psi}{\partial r}\Big|_{r=a} = 0$$

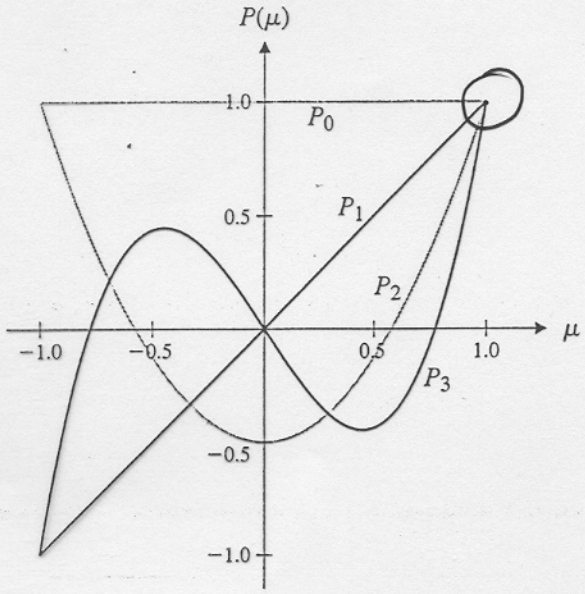
$$\frac{\partial\psi}{\partial r} = A_{10} Y_{10} + \sum_{l,m} B_{lm} \frac{-(l+1)}{r^{l+2}} Y_{lm}$$

$$\frac{\partial\psi}{\partial r}\Big|_{r=a} = 0 = A_{10} Y_{10} - \sum_{l,m} B_{lm} \frac{(l+1)}{a^{l+2}} Y_{lm}$$

For $l=1, m=0$: $\quad A_{10} - B_{10}\frac{(2)}{a^3} = 0 \qquad B_{10} = \frac{a^3}{2} A_{10}$

$$= -\frac{a^3}{2}\frac{v_0\cos\theta}{Y_{10}}$$

For $(l,m) \neq (1,0)$: $B_{lm} = 0$



$P_\ell(1) = 1$

$P_\ell(x)$ = $\cos\theta$

FIGURE 8.3. The first four Legendre polynomials. $P_0(\mu)$ and $P_2(\mu)$ are even functions; $P_1(\mu)$ and $P_3(\mu)$ are odd functions. All the functions are chosen to have the value 1 at $\mu = 1$ (top right).

Lea

P_ℓ^m

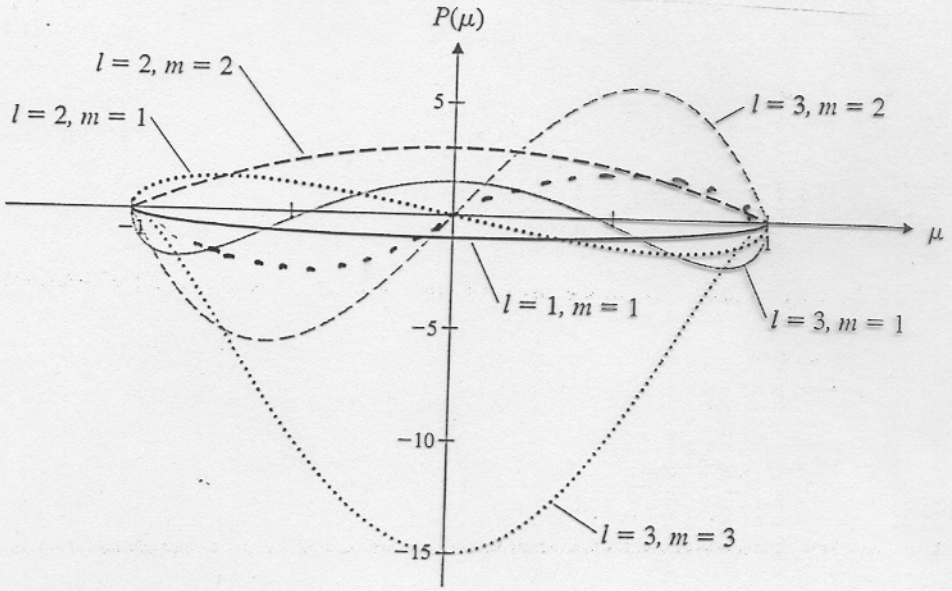


FIGURE 8.7. The first few associated Legendre functions P_l^m. For functions with $m = 0$, see Figure 8.3. Notice that $P_l^m(\pm 1) = 0$ for $m \neq 0$.

Lea

Factor $(-1)^m$ relative to Wyld

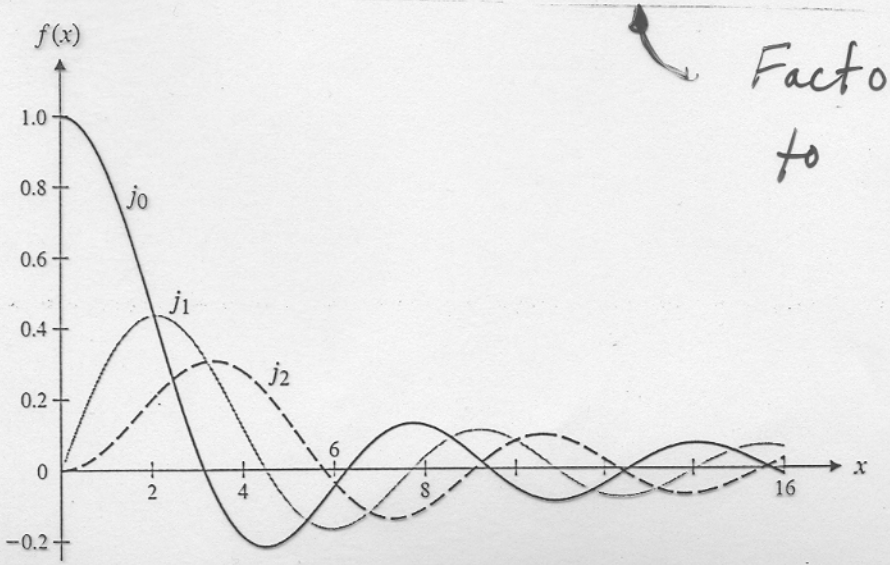


FIGURE 8.17. The first few spherical Bessel functions $j_0(x)$, $j_1(x)$, and $j_2(x)$. Compare with

Lea

$$\psi(r,\theta,\varphi) = c - v_0 r\cos\theta - \frac{a^3 v_0 \cos\theta}{2r^2}$$

Neglecting the arbitrary constant c,

$$\psi(r,\theta,\varphi) = -v_0\left[r + \frac{a^3}{2r^2}\right]\cos\theta$$

$$\vec{v} = -\vec{\nabla}\psi = v_0\left[1 - \frac{a^3}{r^3}\right]\cos\theta\,\hat{r} - v_0\left[1 + \frac{a^3}{2r^3}\right]\sin\theta\,\hat{\theta}$$

where we have used $\vec{\nabla}$ in spherical coordinates:

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

We must also consider the radial dependence for the Helmholtz equation $\nabla^2 u + k^2 u = 0$. This leads us to study spherical Bessel functions. (Ch. 6 Wyld)

With $u = R(r)\,Y_{lm}(\theta,\varphi)$, the radial part of the Helmholtz equation can be put into a Bessel equation form (pg 40-43). We have

$$\frac{1}{r}\frac{d^2}{dr^2}(rR) + k^2 R - \frac{l(l+1)}{r^2}R = 0$$

Note: $\frac{1}{r}\frac{d^2}{dr^2}(rR) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)$

S-L form $\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2 r^2 R - l(l+1)R = 0$

We can put this into the form of Bessel's eqn by defining $R(r) = \frac{1}{\sqrt{r}} S(r)$. The S-L form

becomes, after grouping terms,

$$\underbrace{r^{1/2}\frac{dS}{dr} + r^{3/2}\frac{d^2S}{dr^2}}_{r^{1/2}\left[\frac{dS}{dr} + r\frac{d^2S}{dr^2}\right]} + k^2 r^{3/2} S - \underbrace{\left[\frac{1}{4} + l(l+1)\right]}_{(l+\frac{1}{2})^2}\frac{S}{r^{1/2}} = 0$$

$$\frac{d}{dr}\left(r\frac{dS}{dr}\right)$$

Multiplying by $r^{-1/2}$:

$$\frac{d}{dr}\left(r\frac{dS}{dr}\right) + k^2 r S - (l+\tfrac{1}{2})^2 \frac{S}{r} = 0$$

This is Bessel's equation of order $(l+\frac{1}{2})$. The solution is

$$S(r) = A J_{l+\frac{1}{2}}(kr) + B N_{l+\frac{1}{2}}(kr)$$

Thus, $$R(r) = A\frac{J_{l+\frac{1}{2}}(kr)}{\sqrt{r}} + B\frac{N_{l+\frac{1}{2}}(kr)}{\sqrt{r}}$$

$$= a\sqrt{\frac{\pi}{2kr}}\, J_{l+\frac{1}{2}}(kr) + b\sqrt{\frac{\pi}{2kr}}\, N_{l+\frac{1}{2}}(kr)$$

$j_l(kr) \equiv$ $\qquad\qquad \equiv n_l(kr)$

This defines the spherical Bessel and Neumann fns:

$$j_\ell(kr) = \sqrt{\frac{\pi}{2kr}}\, J_{\ell+\frac{1}{2}}(kr)$$

$$n_\ell(kr) = \sqrt{\frac{\pi}{2kr}}\, N_{\ell+\frac{1}{2}}(kr)$$

Recall the series expansion for $J_m(x)$

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{m+2n}.$$

This yield a series expansion for $j_\ell(x)$

$$j_\ell(x) = \frac{\sqrt{\pi}}{(2)} \sqrt{\frac{2}{x}}\, J_{\ell+\frac{1}{2}}(x)$$

$$= \frac{\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\ell+\frac{1}{2}+1)} \left(\frac{x}{2}\right)^{(\ell+\frac{1}{2})+2n-\frac{1}{2}}$$

$$= \frac{\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\ell+\frac{3}{2})} \left(\frac{x}{2}\right)^{\ell+2n}$$

We can use this to determine $j_0(x)$ in a nice closed form and then use Bessel function recursion relations to find $j_\ell(x)$.

$$j_0(x) = \frac{\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\frac{3}{2})} \left(\frac{x}{2}\right)^{2n}$$

$$\Gamma(n+\tfrac{3}{2}) = (n+\tfrac{1}{2})\,\Gamma(n+\tfrac{1}{2}) = (n+\tfrac{1}{2})(n-\tfrac{1}{2})\,\Gamma(n-\tfrac{1}{2})$$

$$= (n+\tfrac{1}{2})(n-\tfrac{1}{2})(n-\tfrac{3}{2}) \cdots\cdots (\tfrac{1}{2})\,\Gamma(\tfrac{1}{2})$$

$$= (2n+1)(2n-1)(2n-3) \cdots\cdots (1)\left(\tfrac{1}{2}\right)^{n+1}\Gamma(\tfrac{1}{2})$$

$$= \frac{(2n+1)!!}{2^{n+1}}\,\Gamma(\tfrac{1}{2}) = \frac{(2n+1)!!}{2^{n+1}}\sqrt{\pi}$$

$$= \frac{(2n+1)!}{(2n)!!}\,\frac{\sqrt{\pi}}{2^{n+1}} = \frac{(2n+1)!}{(2)^n\,n!}\,\frac{\sqrt{\pi}}{2^{n+1}} = \frac{\sqrt{\pi}\,(2n+1)!!}{n!\,2^{2n+1}}$$

$$j_0(x) = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{n!\,2^{2n+1}}{\sqrt{\pi}\,(2n+1)!}\right] \frac{x^{2n}}{(2)^{2n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\,x^{2n}}{(2n+1)!} = \frac{\sin x}{x}$$

$$\boxed{j_0(x) = \frac{\sin x}{x}}$$

We can prove a recursion relation for the J's which allows us to determine all j_ℓ from j_0.

$$J_m(x) = (x/2)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+m+1)} \frac{(x^2/2)^n}{2^n}$$

Note that $\frac{d}{d(x^2/2)} = \frac{dx}{d(x^2/2)} \frac{d}{dx} = \frac{1}{x} \frac{d}{dx}$

Apply the operator $\frac{1}{x}\frac{d}{dx}$ ℓ times to the ratio $\frac{J_m(x)}{(x/2)^m}$:

$$\left[\frac{1}{x}\frac{d}{dx}\right]^\ell \left(\frac{J_m(x)}{(x/2)^m}\right) = \sum_{n=\ell}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+m+1)\,2^n} (n)(n-1)\cdots(n-\ell+1)\left(\frac{x^2}{2}\right)^{n-\ell}$$

$$= \sum_{n=\ell}^{\infty} \frac{(-1)^n}{(n-\ell)!\,\Gamma(n+m+1)} \frac{(x^2/2)^{n-\ell}}{2^n}$$

To start the sum at 0, put $s = n-\ell$

$$\left[\frac{1}{x}\frac{d}{dx}\right]^\ell \left(\frac{J_m(x)}{(x/2)^m}\right) = \sum_{s=0}^{\infty} \frac{(-1)^{\ell+s}}{s!\,\Gamma(s+\ell+m+1)} \frac{(x^2/2)^s}{2^{\ell+s}}$$

$$= \frac{(-1)^\ell}{2^\ell} \sum_{s=0}^{\infty} \frac{(-1)^s\,(x^2/2)^s}{\Gamma(s+\ell+m+1)\,2^s}$$

Comparing to first line

$$= \frac{(-1)^\ell}{2^\ell} \frac{J_{\ell+m}(x)}{(x/2)^{\ell+m}}$$

So $$\frac{J_{\ell+m}(x)}{x^m} = (-x)^\ell \left[\frac{1}{x}\frac{d}{dx}\right]^\ell \left(\frac{J_m(x)}{x^m}\right)$$

Put $m = 1/2$

$$\frac{J_{\ell+\frac{1}{2}}(x)}{\sqrt{x}} = \sqrt{\frac{2}{\pi}}\, j_\ell(x) = (-x)^\ell \left[\frac{1}{x}\frac{d}{dx}\right]^\ell \underbrace{\left(\frac{J_{1/2}(x)}{\sqrt{x}}\right)}$$

$$= \sqrt{\frac{2}{\pi}}\, j_0(x)$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{\sin x}{x}$$

$$j_\ell(x) = (-x)^\ell \left[\frac{1}{x}\frac{d}{dx}\right]^\ell \left(\frac{\sin x}{x}\right)$$

The first couple spherical Bessel fns are:

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x$$

Generally, $j_n(x) = f_n(x)\cos x + g_n(x)\sin x$

f_n, g_n: polynomials in $\left(\frac{1}{x}\right)$

For the small x dependence of $j_\ell(x)$, revert to the series expansion on pg 204:

$$j_\ell(x) = \frac{\sqrt{\pi}}{2}\left(\frac{x}{2}\right)^\ell \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\ell+\frac{3}{2})}\left(\frac{x}{2}\right)^{2n}$$

$$\xrightarrow[x\to 0]{} \frac{\sqrt{\pi}}{2}\,\frac{x^\ell}{2^\ell}\,\frac{1}{\Gamma(\ell+\frac{3}{2})} \qquad n=0 \text{ term}$$

From pg 205 $\Gamma(\ell+\frac{3}{2}) = \dfrac{\sqrt{\pi}\,(2\ell+1)!}{\ell!\,2^{2\ell+1}}$

So $j_\ell(x) \xrightarrow[x\to 0]{} \dfrac{2^\ell \ell!}{(2\ell+1)!}\,x^\ell$

Except for $j_0(x)$, all $j_\ell(x=0)$ vanish.

$$j_0(x) = \frac{\sin x}{x} \quad \Rightarrow \quad j_0(x=1) = 1$$

The spherical Neumann function

$$n_\ell(x) = \sqrt{\frac{\pi}{2x}}\,N_{\ell+\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}\,\frac{\left[J_{\ell+\frac{1}{2}}(x)\cos\left[(\ell+\tfrac{1}{2})\pi\right] - J_{-(\ell+\frac{1}{2})}(x)\right]}{\sin\left[(\ell+\frac{1}{2})\pi\right]}$$

($=0$ for integer ℓ)

$$= (-1)^{\ell+1}\sqrt{\frac{\pi}{2x}}\,J_{-\ell-\frac{1}{2}}(x)$$

(Note typo in Wyld pg 180 refers to (4.2.19) → should be (4.3.2))

(Details in Wyld)

Inserting series expansion for J yields

$$n_0(x) = -\frac{\cos x}{x}$$

Another recursion relation leads to

$$n_\ell(x) = -(-x)^\ell \left[\frac{1}{x}\frac{d}{dx}\right]^\ell \left[\frac{\cos x}{x}\right].$$

The next two n_ℓ are:

$$n_1(x) = -\frac{\sin x}{x} - \frac{\cos x}{x^2}$$

$$n_2(x) = -\frac{3}{x^2}\sin x - \left[\frac{3}{x^3} - \frac{1}{x}\right]\cos x$$

Generally,

$$n_\ell(x) \sim -g_\ell(x)\cos x - f_\ell(x)\sin x$$

The small x limit is:

$$n_\ell(x) \xrightarrow[x\to 0]{} -\frac{(2\ell)!}{2^\ell \ell!}\frac{1}{x^{\ell+1}}$$

Just like the Neumann fns, the spherical Neumann fns are divergent at $x=0$.

The spherical Hankel functions are defined just as the ordinary Hankel functions are.

$$h_\ell^{(1)}(x) = j_\ell(x) + i\, n_\ell(x) = \sqrt{\frac{\pi}{2x}}\, H^{(1)}_{\ell+\frac{1}{2}}(x)$$

$$h_\ell^{(2)}(x) = j_\ell(x) - i\, n_\ell(x) = \sqrt{\frac{\pi}{2x}}\, H^{(2)}_{\ell+\frac{1}{2}}(x)$$

Using $j_\ell(x) = (-x)^\ell \left[\frac{1}{x}\frac{d}{dx}\right]^\ell \left[\frac{\sin x}{x}\right]$ and

$$n_\ell(x) = -(-x)^\ell \left[\frac{1}{x}\frac{d}{dx}\right]^\ell \left[\frac{\cos x}{x}\right],$$ we see.

$$h_\ell^{(1)}(x) = (-x^\ell)\left[\frac{1}{x}\frac{d}{dx}\right]^\ell \underbrace{\left[\frac{\sin x - i\cos x}{x}\right]}_{\frac{e^{ix}}{ix}}$$

$$h_\ell^{(2)}(x) = (-x)^\ell \left[\frac{1}{x}\frac{d}{dx}\right]^\ell \left[\frac{e^{-ix}}{-ix}\right]$$

- Rayleigh formulas

The asymptotic limits for these functions ($x \to \infty$) are very useful (scattering problems). They are most easily seen from that limiting behaviour of the Hankel functions.

We had $$H_m^{(1)}(x) \xrightarrow[x\to\infty]{} \sqrt{\frac{2}{\pi x}} \exp\left[i\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)\right]$$

$$H_m^{(2)}(x) \xrightarrow[x\to\infty]{} \sqrt{\frac{2}{\pi x}} \exp\left[-i\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)\right]$$

so we find

$$h_l^{(1)}(x) = \sqrt{\frac{\pi}{2x}}\, H_{l+\frac{1}{2}}^{(1)}(x) \xrightarrow[x\to\infty]{} \frac{\exp\left[i\left(x - (l+1)\frac{\pi}{2}\right)\right]}{x} = -i\frac{e^{i(x - l\pi/2)}}{x}$$

$$h_l^{(2)}(x) = \sqrt{\frac{\pi}{2x}}\, H_{l+\frac{1}{2}}^{(2)}(x) \xrightarrow[x\to\infty]{} \frac{\exp\left[-i\left(x - (l+1)\frac{\pi}{2}\right)\right]}{x} = i\frac{e^{-i(x - l\pi/2)}}{x}$$

It immediately follows that

$$j_l(x) \xrightarrow[x\to\infty]{} \frac{1}{x}\cos\left[x - (l+1)\frac{\pi}{2}\right] = -\frac{1}{x}\cos\left(x - l\frac{\pi}{2}\right)$$

$$n_l(x) \xrightarrow[x\to\infty]{} \frac{1}{x}\sin\left[x - (l+1)\frac{\pi}{2}\right]$$

The orthogonality properties of the spherical Bessel functions j_ℓ follow directly from the S-L perspective, and from our earlier results for Bessel functions J_ℓ.

In S.L. form, we had

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \underbrace{k^2}_{\lambda}\, r^2 R - \ell(\ell+1) R = 0$$

(r^2: weight w)

with solutions $R \sim j_\ell(kr)$.

Thus, for R obeying S-L b.c.'s $\left[\alpha_1 R + \beta_1 \frac{dR}{dx} = 0\right]\Big|_{x=x_1}$ $\left[\alpha_2 R + \beta_2 \frac{dR}{dx} = 0\right]\Big|_{x=x_2}$ we expect an orthogonality relation. Here, for range $a \le r \le b$:

$$\int_a^b dr\; \underbrace{r^2}_{w}\, j_\ell(\underbrace{kr}_{k_{\ell n}})\, j_\ell(\underbrace{k'r}_{k_{\ell n'}}) = 0 \quad \text{for } k \neq k'$$

Let's consider the range $0 \longrightarrow b$. We get orthogonality relations from

$$(y_n, \mathcal{L} y_m) - (y_m, \mathcal{L} y_n).$$

Upon integrating by parts, there's a factor $w(x)$ in the boundary terms. So for $w(r) = r^2$, this vanishes at $r=0$. So we do not have to specify b.c. for $j_\ell(kr)$ at $r=0$, other than that it is finite.

We then determine $\int_0^b dr\, r^2 \left[j_\ell (k_{\ell n} r) \right]^2$ upon specifying a b.c. at $r = b$.

Assume b.c. is $j_\ell (kb) = 0$. Then k must be such that $k_{\ell n} b = z_{\ell n}$ = nth zero of $j_\ell (x)$.

= nth zero of $J_{\ell+\frac{1}{2}}(x)$.

$$\int_0^b dr\, r^2 \left[j_\ell (k_{\ell n} r) \right]^2 =$$

$$= \frac{\pi}{2 k_{\ell n}} \int_0^b dr \underbrace{\frac{r^2}{r}} \left[J_{\ell+\frac{1}{2}} (k_{\ell n} r) \right]^2 = \frac{\pi}{2 k_{\ell n}} \frac{b^2}{2} \left[J'_{\ell+\frac{1}{2}} (k_{\ell n} b) \right]^2$$

$w = r$ is the weight fn. for Bessel fns.

Since $j_\ell (x) = \sqrt{\frac{\pi}{2x}}\, J_{\ell+\frac{1}{2}} (x)$,

$$j'_\ell (x) = \frac{d j_\ell (x)}{dx} = \sqrt{\frac{\pi}{2}} \left[-\frac{1}{2x^{3/2}} J_{\ell+\frac{1}{2}} (x) + \frac{1}{\sqrt{x}} J'_{\ell+\frac{1}{2}} (x) \right]$$

Evaluating this at the nth zero of $J_{\ell+\frac{1}{2}}$:

$$j'_\ell (z_n) = \sqrt{\frac{\pi}{2 z_n}}\, J'_{\ell+\frac{1}{2}} (z_n)$$

So, in terms of the derivative of the spherical Bessel function:

$$\int_0^b dr\; r^2 \left[j_\ell(k_{\ell n} r) \right]^2 = \frac{b^3}{2} \left[j_\ell'(k_{\ell n} b) \right]^2$$

Then we can expand a function of r, f(r), in terms of the complete set of eigenfns, the spherical Bessel fns.

$$f(r) = \sum_{n=1}^{\infty} c_n\, j_\ell(k_{\ell n} r)$$

The coefficients in the expansion are obtained using the orthogonality of the j_ℓ.

$$c_n = \frac{\int_0^b dr\; r^2\, f(r)\, j_\ell(k_{\ell n} r)}{\int_0^b dr\; r^2 \left[j_\ell(k_{\ell n} r) \right]^2}$$

If we have an infinite range $b \to \infty$, we go from the discrete zeros $k_{\ell m} b$ to a continuous parameter k. We can use the Bessel fn results to get the appropriate relations for spherical Bessel functions.

We had (pg 126):

$$\int_0^a d\rho\, \rho\, J_m(k_{mn}\rho)\, J_m(k_{mn'}\rho) = \frac{a^2}{2}\left[J_m'(k_{mn}a)\right]^2 \delta_{nn'}$$

$\downarrow$ $a \to \infty$

$$\int_0^\infty d\rho\, \rho\, J_m(k\rho)\, J_m(k'\rho) = \frac{\delta(k-k')}{k}$$

We want $\int_0^\infty dr\, r^2\, j_\ell(kr)\, j_\ell(k'r)$

$$= \frac{\pi}{2}\frac{1}{\sqrt{kk'}}\int_0^\infty dr\, r\, J_{\ell+\frac{1}{2}}(kr)\, J_{\ell+\frac{1}{2}}(k'r)$$

$$= \frac{\pi}{2\sqrt{kk'}}\,\frac{\delta(k-k')}{k}$$

Thus $$\boxed{\int_0^\infty dr\, r^2\, j_\ell(kr)\, j_\ell(k'r) = \frac{\pi}{2k^2}\,\delta(k-k')}$$

orthogonality relation on infinite range.

For the infinite range case, the series expansion of a function $f(r)$ in spherical Bessel fns becomes an integral over the parameter k.

Conventionally write C_n

$$f(r) = \int_0^\infty dk\ k^2\ F(k)\ j_\ell(kr) \qquad *$$

The "coefficient" $F(k)$ is determined as follows:

$$\int_0^\infty dr\ r^2\ j_\ell(k'r)\ f(r) = \int_0^\infty dk\ k^2 \underbrace{\int_0^\infty dr\ r^2\ j_\ell(k'r)\ j_\ell(kr)}_{\frac{\pi}{2}\frac{\delta(k-k')}{k^2}} F(k)$$

$$= \frac{\pi}{2} F(k')$$

Thus $$F(k) = \frac{2}{\pi} \int_0^\infty dr'\ r'^2\ j_\ell(kr')\ f(r')$$

Putting this expression for $F(k)$ back into * yields a closure relation for the j_ℓ.

$$f(r) = \int_0^\infty dr'\ r'^2 \left[\int_0^\infty dk\ k^2\ j_\ell(kr)\ j_\ell(kr') \frac{2}{\pi} \right] f(r')$$

Identify $$\frac{2}{\pi}\int_0^\infty dk\ k^2\ j_\ell(kr)\ j_\ell(kr') = \frac{\delta(r-r')}{r^2}$$

Expansion of "waves" in spherical coordinates

It is often appropriate to describe the emission, absorption or scattering of waves of some form in spherical coordinates. So express waves as superposition of the eigenfunctions of the wave equation → Helmholtz equation is the spatial part.

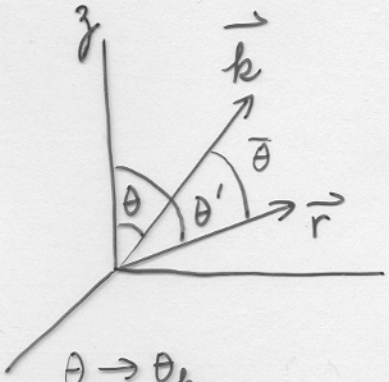
Previously we derived the generating fn for ordinary Bessel fns by using the fact that the Helmholtz eqn has a plane wave soln and expressing that solution as a superposition over cylindrical coordinate eigenfns, namely

$$e^{iky} = e^{ikp\sin\varphi} = \sum_{m=-\infty}^{+\infty} e^{im\varphi} J_m(kp)$$

Now we wish to express both plane wave and spherical wave forms in terms of spherical coordinates eigenfns.

Consider plane wave first $e^{i(\vec{k}\cdot\vec{r} - \omega t)}$.

The spatial part $e^{i\vec{k}\cdot\vec{r}}$ is a solution of the Helmholtz equation $(\nabla^2 + |\vec{k}|^2)\, u(\vec{r}) = 0$



$\theta \to \theta_k$

$\theta' \to \theta_r$

$$e^{i\vec{k}\cdot\vec{r}} = e^{ikr\cos\bar{\theta}}$$

$$= \sum_{l=0}^{\infty} c_l j_l(kr) P_l(\cos\bar{\theta})$$

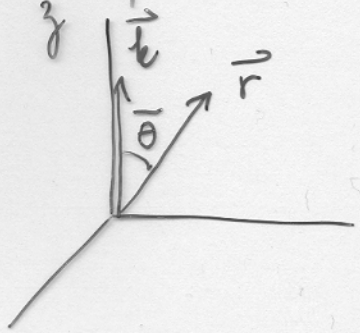
eigenfns of Helmholtz eq. in spherical coords

(P_l rather than Y_{lm} since $e^{i\vec{k}\cdot\vec{r}}$ depends only on $\bar{\theta}$)

by spherical harmonics addition theorem:

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{l=0}^{\infty} c_l j_l(kr) \frac{4\pi}{2l+1} \sum_{m=-l}^{+l} Y_{lm}^*(\theta_k, \varphi_k) Y_{lm}(\theta_r, \varphi_r)$$

We must determine the c_l. Working in a space with $\hat{z}$ axis aligned with $\vec{k}$:



$$e^{i\vec{k}\cdot\vec{r}} = e^{ikz} = e^{ikr\cos\bar{\theta}} = e^{ikr\cos\theta}$$

Drop $\bar{\theta}$ for convenience.

In this coord system, $\bar{\theta}$ is just the polar angle and there is no azimuthal angle dependence.

$$e^{i\vec{k}\cdot\vec{r}} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} c_l j_l(kr) P_l(\cos\theta)$$

Use orthogonality of P_l. Multiply by $P_n(\cos\theta)$ and integrate over $\cos\theta$.

$$\int_{-1}^{+1} d\cos\theta\, P_n(\cos\theta)\, e^{ikr\cos\theta} = \sum_{l=0}^{\infty} c_l\, j_l(kr) \underbrace{\int_{-1}^{+1} d\cos\theta\, P_n(\cos\theta) P_l(\cos\theta)}_{\frac{2}{2n+1}\delta_{ln}}$$

Thus

$$c_l\, j_l(kr) = \frac{2l+1}{2} \underbrace{\int_{-1}^{+1} d\cos\theta\, e^{ikr\cos\theta} P_l(\cos\theta)}_{\text{Evaluate this integral.}}$$

$$I_l = \int_{-1}^{+1} dx\, e^{ikrx} \underbrace{P_l(x)}_{\text{Rodrigues} = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l}$$

u ↑ , $dv \longrightarrow v \sim \frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l$

Integrate by parts l times. With less than l derivatives, the $(x^2-1)^l$ term will have at least one factor (x^2-1) in the boundary terms. Thus these terms vanish. This leaves:

$$I_l = \frac{(-ikr)^l}{2^l l!} \underbrace{\int_{-1}^{1} dx\, e^{ikrx} (x^2-1)^l}_{\equiv I_l \quad \text{Evaluate by integration by parts.}}$$

Integrating by parts, we find:

$$I_\ell = \int_{-1}^{1} dx\; e^{ikrx}(x^2-1)^\ell = +\frac{2\ell}{kr}\int_{-1}^{1} dx\; ixe^{ikrx}(x^2-1)^{\ell-1}$$

notice $\rightarrow$

$$= \frac{2\ell}{kr}\frac{d}{d(kr)}\underbrace{\int_{-1}^{1} dx\; e^{ikrx}(x^2-1)^{\ell-1}}_{=I_{\ell-1}}$$

$$I_\ell = \frac{2\ell}{kr}\left[\frac{d}{d(kr)}\right] I_{\ell-1} = \frac{2\ell}{kr}\,\frac{2(\ell-1)}{kr}\left[\frac{d}{d(kr)}\right] I_{\ell-2} \cdots\cdots$$

$$\therefore\; I_{\ell-1} = \frac{2(\ell-1)}{kr}\,\frac{d}{d(kr)}\, I_{\ell-2}$$

Finally $I_\ell = 2^\ell\, \ell!\left[\frac{1}{kr}\frac{d}{d(kr)}\right]^\ell \underbrace{\int_{-1}^{1} dx\; e^{ikrx}}_{\frac{2\sin kr}{kr}}$

(ℓ steps)

So $I_\ell = 2^{\ell+1}\,\ell!\left[\frac{1}{kr}\frac{d}{d(kr)}\right]^\ell\left(\frac{\sin kr}{kr}\right)$

Thus $c_\ell\, j_\ell(kr) = (2\ell+1)\, i^\ell\,(-kr)^\ell\left[\frac{1}{kr}\frac{d}{d(kr)}\right]^\ell\left(\frac{\sin kr}{kr}\right)$

Recall $j_\ell(x) = (-x)^\ell \left[\frac{1}{x}\frac{d}{dx}\right]^\ell \left[\frac{\sin x}{x}\right]$.

So we can identify the coefficients

$$c_\ell = (2\ell+1)\, i^\ell$$

Thus,

$$e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1)\, j_\ell(kr)\, P_\ell(\cos\theta)$$

Bauer formula

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell=0}^{\infty} i^\ell\, 4\pi\, j_\ell(kr) \sum_{m=-\ell}^{+\ell} Y_{\ell m}^{*}(\theta_k, \varphi_k)\, Y_{\ell m}(\theta_r, \varphi_r)$$

This expresses a plane wave in spherical coordinates. Will apply to scattering a plane wave from a sphere. But first, consider spherical waves.

3-d Wave equation: $\nabla^2 \psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$

Separating time and spatial dependence

$\psi(\vec{r}, t) = u(\vec{r})\, T(t)$ leads to Helmholtz eq

for spatial part $\nabla^2 u(\vec{r}) + k^2 u(\vec{r}) = 0$

$k^2 = |\vec{k}|^2$

Time part $\frac{d^2T}{dt^2} = -(kc)^2 T = -\omega^2 T$

has solutions $T(t) = e^{\pm i\omega t}$

In Cartesian coordinates, linear combinations of $\left[u(\vec{r}) = e^{\pm i\vec{k}\cdot\vec{r}}\right] \cdot \left[T(t) = e^{\pm i\omega t}\right]$ give

standing waves: $(\cos \vec{k}\cdot\vec{r})\ T(t)$

$(\sin \vec{k}\cdot\vec{r})\ T(t)$

↑ plane

For standing waves we have nodal surfaces (planes in this case) where $\psi(\vec{r}, t) = 0\ \forall t$.

Alternately we have linear combination that represent travelling plane waves:

$e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ ← moves in $+\vec{k}$ direction

$\cos(\vec{k}\cdot\vec{r} - \omega t)$ ← (or cos(arg); sin(arg))

$e^{i(\vec{k}\cdot\vec{r} + \omega t)}$ ← moves in $-\vec{k}$ direction

Assumes $\omega > 0$.

In spherical coordinates, the solutions, of the Helmholtz equation $u(\vec{r})$ are products of spherical Bessel fns of various types (radial part) and spherical harmonics (angular part).

$$\psi(\vec{r},t) \sim \left[A\, j_\ell(kr) + B\, n_\ell(kr) \right] Y_{\ell m}(\theta,\varphi)\, e^{\pm i\omega t}$$

or $\left[A\, h_\ell^{(1)}(kr) + B\, h_\ell^{(2)}(kr) \right]$

The forms with j_ℓ or n_ℓ provide standing waves (nodal surfaces — spheres with radius such that kr = a zero of the fn.)

For an "interior" problem, the divergent n_ℓ are dropped.

Normal modes problems with a spherical geometry have solutions of the general form:

$$\psi_{\ell mn}(\vec{r},t) = j_\ell(k_{\ell n} r)\, Y_{\ell m}(\theta,\varphi)\, e^{-i\omega_{\ell n} t}$$

↑ conventional sign choice for time dependence

Boundary conditions then dictate the exact solution. For instance, for acoustic waves in a cavity, ψ represents the density or pressure fluctuations (see Wyld Ch1) and a typical boundary condition is that the normal derivative should vanish at the walls. So sound waves in a sphere would have $\left.\frac{\partial \psi}{\partial r}\right|_{r=a} = 0.$

This boundary condition determines the k_{ln}:

$$j_l'(k_{ln} a) = 0$$

That is k_{ln} are such that $k_{ln} a = \tilde{z}_{ln}$, the nth zero of $j_l'(x)$.

Whereas the j_l and n_l are like the trig fns, the $h_l^{(1)}$ and $h_l^{(2)}$ are like the oscillating exponentials $e^{\pm ikr}$. Thus linear combinations of spherical coordinate eigenfns of the spherical Hankel form give travelling waves

$$\psi(\vec{r},t) \sim \begin{cases} h_l^{(1)}(kr)\, Y_{lm}(\theta,\varphi)\, e^{-i\omega t} \\ h_l^{(2)}(kr)\, Y_{lm}(\theta,\varphi)\, e^{-i\omega t} \end{cases}$$

$\omega = kc > 0$

Because we have $h_l^{(1)}(x) \underset{x\to\infty}{\longrightarrow} \frac{e^{ix}}{x} e^{-i(l+1)\pi/2}$

$$h_l^{(2)}(x) \underset{x\to\infty}{\longrightarrow} \frac{e^{-ix}}{x} e^{i(l+1)\pi/2}$$

the asymptotic limits go as

$$\psi(\vec{r},t) \xrightarrow[r\to\infty]{} \left\{ \begin{array}{l} (-i)^{l+1} \dfrac{e^{i(kr-\omega t)}}{kr} Y_{lm}(\theta,\varphi) \\ (i)^{l+1} \dfrac{e^{-i(kr+\omega t)}}{kr} Y_{lm}(\theta,\varphi) \end{array} \right.$$

$e^{i(kr-\omega t)}$ $\longrightarrow$ outgoing spherical wave $h_l^{(1)}$

$e^{-i(kr+\omega t)}$ $\longrightarrow$ incoming $h_l^{(2)}$

These interpretations are exchanged if the $e^{+i\omega t}$ convention for time dependence is adopted.

Consider the radiation of spherical waves by a source $\longrightarrow$ treat the description of the source as a boundary condition at a radius a.

Choose a monochromatic source generating waves such that at $r=a$:

$$\psi(a,\theta,\varphi,t) = \underbrace{F(\theta,\varphi)}_{\text{some given function}} e^{-i\omega t}$$

Express the radiated waves as an expansion in eigenfns: $\psi(r,\theta,\varphi,t) = \sum_{l,m} a_{lm} h_l^{(1)}(kr) Y_{lm}(\theta,\varphi) e^{-i\omega t}$

$(r>a)$

This is a convenient expression to consider the flux of radiated energy and the power emitted per unit solid angle.

$$\text{Energy flux} = \underbrace{c}_{\substack{\uparrow \\ \text{velocity in} \\ \text{wave eqn}}} \cdot (\text{Energy density}) = c\,|\psi|^2 = c\,\frac{|f(\theta,\varphi)|^2}{r^2}$$

So the rate of energy emitted through solid angle $d\Omega$ is: power per unit solid angle =

(Energy flux) x (Area subtended by $d\Omega$)

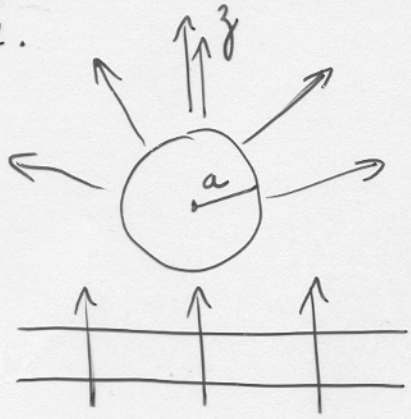
$$= c\,\frac{|f(\theta,\varphi)|^2}{r^2} \times (r^2\, d\Omega)$$

$$\therefore \quad \frac{dP}{d\Omega} = c\,|f(\theta,\varphi)|^2$$

where $$f(\theta,\varphi) = \frac{1}{k}\sum_{l,m} a_{lm}\,(-i)^{l+1}\,Y_{lm}(\theta,\varphi)$$

and $$a_{lm} = \frac{1}{h_l^{(1)}(ka)} \int d\Omega\, F(\theta,\varphi)\, Y_{lm}^{*}(\theta,\varphi)$$

We can apply the results developed above to the scattering of a monochromatic plane wave by a sphere.



Azimuthally symmetric

$$\psi_{inc} = e^{i(kz-\omega t)}$$

incident wave

Unit amplitude chosen

$\omega = kc$

$$\psi_{inc} = e^{ikr\cos\theta}\, e^{-i\omega t}$$

$$= \sum_{l=0}^{\infty} (2l+1)\, i^l\, j_l(kr)\, P_l(\cos\theta)\, e^{-i\omega t}$$

We expect the outgoing wave to be a superposition of the incident wave and an outgoing spherical wave, ψ_{sc}, of the same frequency.

← scattered wave

Generally, $\psi = \psi_{inc} + \psi_{sc}$

will set $m=0$

$$= e^{i(kz-\omega t)} + \sum_{l,m} a_{lm}\, h_l^{(1)}(kr)\, Y_{lm}(\theta,\phi)\, e^{-i\omega t}$$

↑ These coefficients only determined by the specific boundary conditions of an actual problem. But general asymptotic behaviour is known and gives "scattering amplitude".

$$\psi_{sc} \xrightarrow[r\to\infty]{} f(\theta,\varphi)\,\frac{e^{i(kr-\omega t)}}{r}$$

scattering amplitude $\longrightarrow$ $f(\theta,\varphi) = \sum_{l,m} a_{lm} \frac{(-1)^{l+1}}{k} Y_{lm}(\theta,\varphi)$

We define the differential cross section, $d\sigma$, as the ratio of (the flux of ψ_{sc} through $d\Omega$) to (the flux per unit area of incident wave).

$$\frac{d\sigma}{d\Omega} = \frac{c\,|f(\theta,\varphi)|^2}{c\,|e^{i(kz-\omega t)}|^2} = |f(\theta,\varphi)|^2$$

Apply the general format to a particular problem — the scattering of sound waves by a sphere.

Assume that the boundary condition is that, at the surface of the sphere ($r=a$), the velocity of the air molecules has no radial component.

The "sound wave" ψ is related to the velocity as $\vec{v} = -\vec{\nabla}\psi$. The condition of tangential velocity is $\left.\frac{\partial \psi}{\partial r}\right|_{r=a} = 0$.

Setting $m=0$,

$$\psi_{sc} = \sum_l a_l\, h_l^{(1)}(kr)\, P_l(\cos\theta)\, e^{-i\omega t}$$

$$\psi = \psi_{inc} + \psi_{sc}$$

$$= \sum_{l=0}^{\infty} \left[(2l+1) i^l j_l(kr) + a_l h_l^{(1)}(kr) \right] P_l(\cos\theta)\, e^{-i\omega t}$$

$$\left.\frac{\partial \psi}{\partial r}\right|_{r=a} = 0 = \sum_{l=0}^{\infty} \underbrace{\left[(2l+1) i^l k j_l'(ka) + a_l k h_l^{(1)\prime}(ka) \right]}_{\text{The coefficient of each term in } P_l \text{ must vanish.}} P_l(\cos\theta) e^{-i\omega t}$$

$$\therefore \quad a_l = -\frac{(2l+1) i^l j_l'(ka)}{h_l^{(1)\prime}(ka)}$$

$$\psi = \sum_{l=0}^{\infty} (2l+1) i^l \left[j_l(kr) - \frac{j_l'(ka)}{h_l^{(1)\prime}(ka)} h_l^{(1)}(kr) \right] P_l(\cos\theta)\, e^{-i\omega t} \quad (*)$$

We can identify the scattering amplitude $f(\theta,\varphi) = f(\theta)$:

$$f(\theta) = \frac{1}{k} \sum_l a_l (-1)^{l+1} P_l(\cos\theta) \qquad \text{as}$$

$$= \frac{i}{k} \sum_l (2l+1) \frac{j_l'(ka)}{h_l^{(1)\prime}(ka)} P_l(\cos\theta)$$

Introduce the concept of a "phase shift" – appears as a simple phase factor in the term representing an outgoing spherical wave in the asymptotic limit of the wave $\sim \left[e^{2i\delta_l} \left(\frac{e^{ikr}}{r} \right) + c \left(\frac{e^{-ikr}}{r} \right) \right]$

Look at the asymptotic limit of ψ (from ⊛) after reexpressing the factor $\dfrac{i j_\ell'(ka)}{h_\ell^{(1)\prime}(ka)}$.

$$\frac{i j_\ell'(ka)}{h_\ell^{(1)\prime}(ka)} = \frac{i j_\ell'}{[j_\ell' + i n_\ell']} = \frac{1}{[n_\ell'/j_\ell' - i]} \equiv \frac{1}{[\cot\delta_\ell(k) - i]}$$

$$\tan\delta_\ell(k) \equiv \frac{j_\ell'(ka)}{n_\ell'(ka)}$$

This form of the definition of the phase shift comes about because our b.c. was a derivative form. Alternately, you end up with a form $\tan\delta_\ell = j_\ell / n_\ell$.

$$= \frac{\sin\delta_\ell}{\cos\delta_\ell - i\sin\delta_\ell} = e^{i\delta_\ell}\sin\delta_\ell$$

Put this into ⊛ and use the asymptotic limits

$$j_\ell(x) \xrightarrow[x\to\infty]{} \frac{1}{x}\cos[x - (\ell+1)\pi/2]$$

$$h_\ell^{(1)}(x) \xrightarrow[x\to\infty]{} (-i)^{\ell+1}\frac{e^{ix}}{x}$$

$$\psi \xrightarrow[r\to\infty]{} \sum_{\ell=0}^{\infty}(2\ell+1)\left[i^\ell \frac{\cos(kr - (\ell+1)\pi/2)}{kr} + e^{i\delta_\ell(k)}\sin\delta_\ell(k)\frac{e^{ikr}}{kr}\right] P_\ell(\cos\theta)e^{-i\omega t}$$

A little algebra later,

$$\psi \xrightarrow[r\to\infty]{} \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\left[(-1)^{l+1}\frac{e^{-ikr}}{r} + e^{2i\delta_l(k)}\frac{e^{ikr}}{r}\right]P_l e^{-i\omega t}$$

$\longrightarrow$ Quantum "square well"

$e^{2i\delta_l(k)}$: phase of outgoing wave shifted by presence of sphere.

In terms of the phase shift, the scattering amplitude is

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\, e^{i\delta_l}\sin\delta_l\, P_l(\cos\theta)$$

The scattering cross sections can be found, using the orthogonality of the Legendre polynomials:

$$\sigma_{tot} = \int d\Omega \frac{d\sigma}{d\Omega} = \int d\Omega\, |f(\theta)|^2$$

$$= \int d\Omega \left[\frac{1}{k}\sum_{l}(2l+1)e^{i\delta_l}\cdots\right]\left[\frac{1}{k}\sum_{l'}(2l'+1)e^{-i\delta_{l'}}\cdots\right]$$

Since $\int d\Omega\, P_l(\cos\theta)\, P_{l'}(\cos\theta) = \frac{4\pi}{2l+1}\delta_{ll'}$:

$$\sigma_{tot} = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l(k)$$

Note that $\mathrm{Im} f(0)$ is related to σ_{tot} as:

$$\mathrm{Im} f(0) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) \underbrace{\mathrm{Im}(e^{i\delta_l})}_{\sin\delta_l} \sin\delta_l \underbrace{P_l(1)}_{=1}$$

$$= \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) \sin^2\delta_l = \frac{k}{4\pi} \sigma_{tot}$$

This is known as the optical theorem and is useful for calculating cross section from scattering amplitude.

We get a simplified form for the phase shift for certain cases. For instance, assume $ka \ll 1$. This corresponds to small wave number $\longrightarrow$ large wavelength $\rightarrow$ low energy.

In the case we've considered (sound scattering from sphere) $\tan\delta_l = \dfrac{j_l'(ka)}{n_l'(ka)}$. So consider this for $ka \ll 1$. We have the limits for small x:

$$j_l(x) \underset{x\to 0}{\longrightarrow} \frac{2^l l!\, x^l}{(2l+1)!}$$

$$n_l(x) \underset{x\to 0}{\longrightarrow} \frac{-(2l)!}{2^l l!} \frac{1}{x^{l+1}}$$

Differentiating and evaluating at $x = ka$:

$$j_\ell'(x)\Big|_{x=ka} = \frac{2^\ell \ell!\, \ell\, (ka)^{\ell-1}}{(2\ell+1)!}$$

$$n_\ell'(x)\Big|_{x=ka} = +\frac{(2\ell)!\,(\ell+1)}{2^\ell \ell!\,(ka)^{\ell+2}}$$

$$\therefore \quad \tan\delta_\ell(k) \xrightarrow[ka<<1]{} \frac{(2^\ell \ell!)^2\, \ell\, (ka)^{2\ell+1}}{(2\ell)!\,(2\ell+1)!\,(\ell+1)} \ll 1 \qquad (\dagger)$$

(small)

[For $\ell=0$ case, use $j_0(x) = \frac{\sin x}{x}$ and $n_0(x) = -\frac{\cos x}{x}$ directly to find $\tan\delta_0(k) = \frac{j_0'(ka)}{n_0'(ka)} \xrightarrow[ka<<1]{} -\frac{(ka)^3}{3}$]

General ℓ, $\tan\delta_\ell(k) \ll 1$ so $\delta_\ell(k) \ll 1$.

Thus the factor in $f(\theta) \sim e^{i\delta_\ell(k)} \sin\delta_\ell(k) \sim \underbrace{\cos\delta_\ell}_{\sim 1}\ \underbrace{\sin\delta_\ell}_{\sim \delta_\ell}$

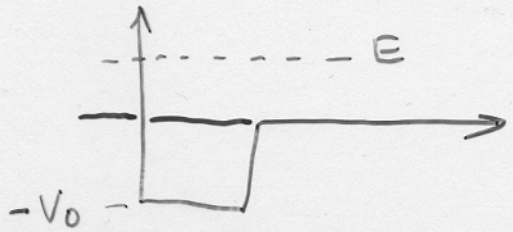
$\sim \delta_\ell \sim \tan\delta_\ell$

$$f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty} (2\ell+1)\tan\delta_\ell\, P_\ell(\cos\theta)$$

$$\tan\delta_0 = -\frac{(ka)^3}{3} \qquad \text{From } (\dagger)\ \tan\delta_1 = \frac{(ka)^3}{6}$$

$$f(\theta) = \frac{1}{k}\left[-\frac{(ka)^3}{3}\underbrace{P_0}_{=1} + 3\frac{(ka)^3}{6}\underbrace{P_1}_{=\cos\theta} + \cdots\right] = -\frac{k^2a^3}{3}\left[1 - \frac{3}{2}\cos\theta - \cdots\right]$$

Continuum or Scattering States



$E > 0$

Equations identical to bound state problem but now $E > 0$.

$r<a$: $\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{l(l+1)}{r^2}R + \frac{2\mu}{\hbar^2}(V_0+E)R = 0$

$r>a$: $\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{l(l+1)}{r^2}R + \frac{2\mu E}{\hbar^2}R = 0$

For $r<a$, must choose regular soln for good behaviour at origin:

$r<a$: $R_l(r) = A\, j_l(Kr)$ $\qquad \boxed{K^2 = \frac{2\mu}{\hbar^2}(V_0+E)}$

$r>a$: $R_l(r) = B\, j_l(kr) + C\, n_l(kr)$ * $\qquad \boxed{k^2 = \frac{2\mu E}{\hbar^2}}$

Matching at $r = a$:

$$K\left[\frac{d j_l(\rho)/d\rho}{j_l(\rho)}\right]\Bigg|_{\rho=Ka} = k\left[\frac{B\, d j_l(\rho)/d\rho + C\, d n_l(\rho)/d\rho}{B\, j_l(\rho) + C\, n_l(\rho)}\right]_{\rho=ka}$$

* If $V=0 \Rightarrow C=0$.

If one manipulates the above to isolate B/C, see that B/C is real since the j's, n's and their derivatives are real. This is useful in the following. Consider the asymptotic (r large) behaviour of the $r>a$ solution:

$$R_\ell(r) \to \frac{B}{kr}\sin\left(kr-\frac{\ell\pi}{2}\right) - \frac{C}{kr}\cos\left(kr-\ell\pi/2\right)$$

$$= \frac{B}{2ikr}\left[e^{i(kr-\ell\pi/2)} - e^{-i(kr-\ell\pi/2)}\right]$$

$$- \frac{C}{2kr}\left[e^{i(kr-\ell\pi/2)} + e^{-i(kr-\ell\pi/2)}\right]$$

Combine like exponentials

$$R_\ell(r) \sim \frac{(-C+iB)}{2kr}\left[e^{-i(kr-\ell\pi/2)} - \underbrace{\frac{(C+iB)}{(-C+iB)}}\, e^{i(kr-\ell\pi/2)}\right]$$

incoming ↑ outgoing ↑

$$\left|\frac{C+iB}{-C+iB}\right|^2 = 1$$

follows from reality of B/C.

Call $\dfrac{C+iB}{-C+iB} \equiv + e^{2i\delta_\ell}$ $\qquad (C=0:\ e^{2i\delta_\ell}=1)$

$(C/B = -\tan\delta_\ell)$

Then

$$R_\ell(r) \sim \frac{(-C+iB)}{2kr}\left[e^{-i\theta} - e^{2i\delta_\ell} e^{i\theta}\right]$$

$\theta = kr - \ell\pi/2$

$$= \frac{(-C+iB)}{2kr}\, e^{i\delta_\ell}\left[e^{-i(\theta+\delta_\ell)} - e^{i(\theta+\delta_\ell)}\right]$$

$$R_\ell(r) \sim N \sin\left(kr - \ell\pi/2 + \underbrace{\delta_\ell(k)}_{\text{phase shift of the } \ell\text{th partial wave}}\right)$$

Phase shift measures displacement in kr relative to free particle ($V=0$).

Scattering characterized by phase shifts.

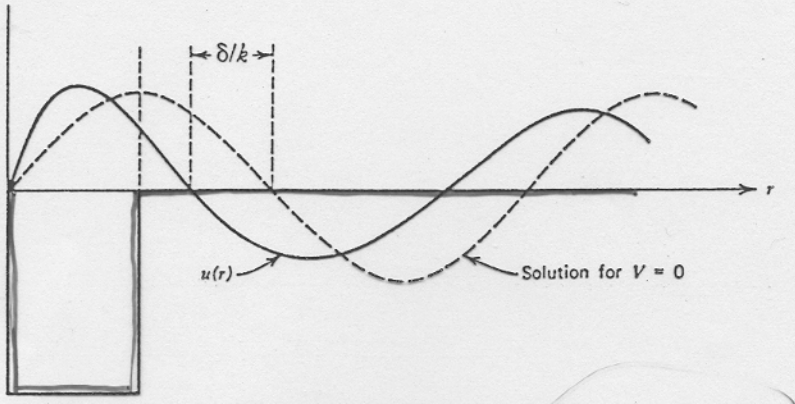


Figure 10-3. Continuum solution $u(r) = rR(r)$ for attractive potential ($l = 0$).

V "draws in" wave fn.

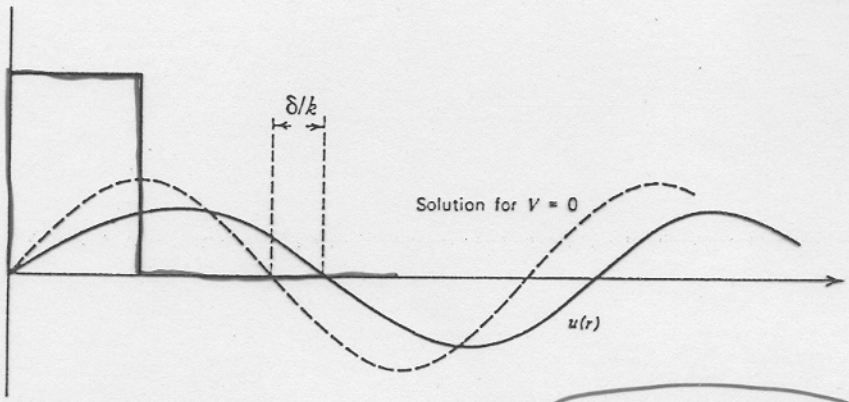


Figure 10-4. Continuum solution $u(r) = rR(r)$ for repulsive potential ($l = 0$).

V "pushes out" wave fn.