The Sturm Liouville Eigenvalue Problem

We have repeatedly seen that we can separate pde's into a set of 2 or more ODE's which, when subject to boundary conditions, become eigenvalue equations whose solutions (eigenfunctions) correspond to characteristic values of the separation constants (eigenvalues). The soln of the pde is an eigenfn expansion, with expansion coefficients determined by b.c.'s.

We have considered the general class of linear second order differential operator $\mathcal{L}$ such that

$$\mathcal{L}\, y(x) = p_0(x)\, y'' + p_1(x)\, y' + p_2(x)\, y$$

So $$\mathcal{L} = p_0(x) \frac{d^2}{dx^2} + p_1(x) \frac{d}{dx} + p_2(x)$$

The homogeneous form of our eigenvalue problem is

weight function

$$\mathcal{L}\, y = -\lambda\, w(x)\, y$$

Assume we study $\mathcal{L}$ on the interval $a \le x \le b$.

Assume p_0 does not vanish within that interval.

Assume p_1 and p_2 are finite in the interval — although a, b may be singular points.

Define the adjoint of the operator $\mathcal{L}$ as $\bar{\mathcal{L}}$:

(Assuming u and v are arbitrary twice differentiable functions)

$$\bar{\mathcal{L}} u \equiv \frac{d^2}{dx^2}(p_0 u) - \frac{d}{dx}(p_1 u) + p_2 u$$

$$= p_0 u'' + (2p_0' - p_1) u' + (p_0'' - p_1' + p_2) u$$

Why do we look at this operator?

Consider the integral:

$$\underbrace{\int_a^b dx\, v\, (\mathcal{L} u)}_{\mathcal{L} \text{ acts on } u} = \int_a^b dx\, v [p_0 u'' + p_1 u' + p_2 u]$$

Integrate by parts to move the derivatives off the function u:

Ignore the boundary terms for now.

$$\int_a^b dx\, v \mathcal{L} u = \int_a^b dx \left[-(v p_0)' u' - (v p_1)' u + v p_2 u\right]$$

$$= \int_a^b dx \left[(p_0 v)'' u - (p_1 v)' u + (p_2 v) u\right]$$

$$= \int_a^b dx\, (\bar{\mathcal{L}} v)\, u$$

$\bar{\mathcal{L}}$ acts on v here.

So $\bar{\mathcal{L}}$ is the operator such that

$$\int_a^b dx\, v \mathcal{L} u = \int_a^b dx\, (\bar{\mathcal{L}} v)\, u$$

Compare the actions of $\mathcal{L}$ and $\bar{\mathcal{L}}$:

$$\mathcal{L}u = \underline{p_0 u''} + p_1 u' + p_2 u$$

$$\bar{\mathcal{L}} u = \underline{p_0 u''} + [2(p_0' - p_1) + p_1] u' + [(p_0'' - p_1') + p_2] u$$

If $p_0' = p_1$ (and, hence, $p_0'' = p_1'$), we will have

$$\mathcal{L}u = \bar{\mathcal{L}} u$$

When an operator and its adjoint have the same action on a funcition, we call it self-adjoint.

In this case,

$$\mathcal{L}u = p_0 u'' + p_0' u' + p_2 u$$

$$= (p_0 u')' + p_2 u \quad = \bar{\mathcal{L}} u$$

This is the so-called Sturm-Liouville operator and we study the eigenvalue problem:

$$\mathcal{L} u = \frac{d}{dx}\left(p(x) \frac{d u(x)}{dx}\right) + q(x)\, u(x) = -\lambda\, w(x)\, u(x)$$

Even if we have an operator $\mathcal{L} = p_0 \frac{d^2}{dx^2} + p_1 \frac{d}{dx} + p_2$ that is not in a self-adjoint form (i.e. $p_0' \neq p_1$), we can relate it to a self-adjoint operator by multiplying it by some appropriate function $h(x)$

Take $\tilde{\mathcal{L}}^* = h\mathcal{L}$ and find h such that

$$(hp_0)' = hp_1$$

$$h'p_0 + hp_0' = hp_1$$

$$\frac{h'}{h} = \frac{(p_1 - p_0')}{p_0}$$

Remember we said p_0 can't be zero.

Integrate this equation to solve for the function h. Then $\tilde{\mathcal{L}}^* = \bar{\tilde{\mathcal{L}}}^*$ is a self-adjoint operator.

So our S-L eigenvalue problem is:

$$\mathcal{L}u = \frac{d}{dx}\left(p\frac{du}{dx}\right) + qu = -\lambda w u$$

separation constant

$w(x) \geq 0$ on the interval $a \leq x \leq b$. w may be zero at isolated points on the interval.

w is called a weight function or density function.

The arguments above motivated the form of the S-L operator. Let's return to the issue of the boundary terms that we dropped.

We'll see that insisting such terms vanish amounts to imposing certain types of boundary conditions on the solns of the S-L problem. This becomes part of the statement of the S-L eigenvalue problem

Let's be more general and assume the functions on which $\mathcal{L}$ acts may be complex.

Define the inner product of two functions as

$$(v, u) = \langle v | u \rangle = \int_a^b dx \; v^*(x)\, u(x)$$

* Modified to include weight fn later.

Go through the details of integrating the inner product of v with $\mathcal{L}u$ by parts, assuming $\mathcal{L}$ takes the S-L form.

$$(v, \mathcal{L}u) = \int_a^b dx \; \underline{v^*} \left[(pu')' + qu) \right]$$

$$= v^*(pu')\Big|_a^b - \int_a^b dx\, \underbrace{v^{*\prime}\, p}\, u' + \int_a^b dx \; v^* q u$$

$$= v^* p u'\Big|_a^b - v^{*\prime} p u\Big|_a^b + \int_a^b dx \; (p v^{*\prime})' u + \int_a^b dx\, v^* q u$$

$$= \int_a^b dx \; ((p v^{*\prime})' + q v^*)\, u + \left[p v^* u' - p v^{*\prime} u \right]\Big|_a^b$$

$$= \int_a^b dx \; (\mathcal{L} v^*)\, u + \left[\;\text{''}\; \right]\Big|_a^b$$

So we find

$$(v, \mathcal{L}u) = \langle v|\mathcal{L}u\rangle = (\mathcal{L}v, u) + \Big[p v^* u' - p v^{*\prime} u \Big]_a^b$$

We can establish that $\langle v|\mathcal{L}u\rangle = \langle \mathcal{L}v|u\rangle$ by choosing boundary conditions such that the last term vanishes. There are various ways to accomplish this.

e.g. $\begin{bmatrix} \alpha_1 u^i(a) + \beta_1 u^{i\prime}(a) = 0 \\ \alpha_2 u^i(b) + \beta_2 u^{i\prime}(b) = 0 \end{bmatrix}$ $\qquad u^i = u, v, \ldots$

<u>or</u> $u^i(a) = u^i(b)$; $u^{i\prime}(a) = u^{i\prime}(b)$; $p(a) = p(b)$

<u>or</u> $p(a)\, u^{i*}(a)\, u^{j\prime}(a) = p(b)\, u^{i*}(b)\, u^{j}(b)$

⋮

Imposing such a boundary condition on the set of functions $u(x)$ defines a function space.

The self adjoint operator $\mathcal{L}$ then defines a <u>Hermitian</u> operator on that space. The operator is Hermitian with respect to those functions obeying the S-L boundary conditions.

The S-L eigenvalue problem is an infinite dimensional analogue of the matrix eigenvalue problem.

constant eigenvalue

$$M u = \lambda u$$

$n \times n$ matrix

$\mathcal{L}$, the S-L operator

n-dimensional column vector (ordered n-tuple of numbers in an n-dimensional vector space)
-eigenvectors of M

$u(x)$, eigenfns of $\mathcal{L}$ corresponding to b.c.'s

There are solutions, u_n, only for certain values of the eigenvalues λ_n.

Suppose M is a Hermitian matrix. That is,

$$M^\dagger = M \qquad \text{where } (M^\dagger)_{ij} = M^*_{ji}$$

$$M_{ij} u_j = \lambda u_i$$

indices indicate the component

$$(M - \lambda \mathbb{1}) u = 0$$

From linear algebra, we know the solution exists only if $\det(M - \lambda \mathbb{1}) = 0$

This is an nth order polynomial equation for λ, the characteristic equation. It has n roots, the λ_i with $1 \le i \le n$. Associated with each root is an eigenvector u^i.

index not the component but rather indicating which eigenvector

We can see that the requirement of M Hermitian implies the λ_i are real.

which eigenvector

$$u^{(i)} \equiv |u^i\rangle = |u^i_1, u^i_2, \ldots, u^i_n\rangle$$

component

Scalar product $\langle u^{(i)} | u^{(j)}\rangle = u^{(i)\dagger} u^{(j)}$

$$= \sum_k u_k^{(i)*} u_k^{(j)}$$

$$u^{\dagger(i)}\left[M u^{(i)} = \lambda_i u^{(i)} \right] \qquad *$$

Take the Hermitian conjugate of $*$:

$$(LHS)^\dagger = u^{(i)\dagger} M^\dagger u^{(i)} = u^{(i)\dagger} M u^{(i)} = \lambda_i^* u^{(i)\dagger} u^{(i)}_{=(RHS)}$$

by Hermiticity

$* - **$: $\quad (\lambda_i - \lambda_i^*) \underbrace{u^{(i)\dagger} u^{(i)}} = 0$

$$= \sum_j u_j^{(i)*} u_j^{(i)} = |u^{(i)}|^2 \neq 0$$

This implies $\lambda_i = \lambda_i^*$ Hermitian matrices have real eigenvalues.

Also recall that eigenvectors corresponding to unequal eigenvalues are orthogonal.

$$\left.\begin{array}{l} u^{(n)} \longrightarrow \lambda_n \\ u^{(m)} \longrightarrow \lambda_m \end{array}\right\} \quad \lambda_n \neq \lambda_m \implies u^{(n)\dagger} u^{(m)} = 0$$

$\boxed{m \neq n}$ $\boxed{\lambda_m \neq \lambda_n}$

Proof:

left multiply

$$u^{(n)\dagger}[\underline{M u^{(m)}} = \lambda_m u^{(m)}] \quad *$$

$$[\ ]^\dagger \longrightarrow \quad [\underline{u^{(n)\dagger} M} = u^{(n)\dagger} \underset{\lambda_n}{\overset{\shortparallel}{\lambda_n^*}}] \cdot \underline{u^{(m)}} \quad **$$

right multiply

$$* - ** \qquad 0 = \underbrace{(\lambda_m - \lambda_n)}_{\neq 0} \underbrace{u^{(n)\dagger} u^{(m)}}_{\therefore\ = 0}$$

For two eigenvectors with the same eigenvalue (degeneracy), any linear combination $a u^{(1)} + b u^{(2)}$ also is an eigenvector with the same eigenvalue. So choose linear combinations such that they are orthogonal. The generalization to an arbitrary number of degenerate eigenvalues is the <u>Gram – Schmidt orthogonalization procedure</u>.

We can also choose to normalize eigenvectors to unity. So, for the <u>matrix</u> eigenvalue problem $M u = \lambda u$ with $\overline{M = M^\dagger}$, the eigenvalues λ_n are all real and their corresponding eigenvectors $u^{(n)}$ are orthonormal

$$u^{(n)\dagger} u^{(m)} = \delta_{nm} \qquad \forall\, n, m$$

Return to S-L theory for Hermitian differential operators $\mathcal{L}\, u(x) + \lambda\, w(x)\, u(x) = 0$

Whereas the eigenvectors of matrix M can be used as basis vectors spanning the vectors space and denoted as $\hat{e}_i$ such that $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$, we now work in a vector space called a function space.

In the usual vector space, any vector $\vec{a}$ can be expressed as

$$\vec{a} = \sum_{i=1}^{n} a_i\, \hat{e}_i \qquad\qquad a_j = \hat{e}_j \cdot \vec{a}$$

components of $\vec{a}$

[Ket notation $\quad |a\rangle = \sum_i a_i |e_i\rangle$

$$a_j = \langle e_j | a\rangle \qquad\qquad \langle e_i | e_j\rangle = \delta_{ij}$$]

When any vector can be so expressed, the basis is said to be complete with respect to the space.

In our function space, each function $u(x)$ is defined on the interval $a \le x \le b$, with the inner product as

$$\langle v | u\rangle = \langle v, u\rangle = \int_a^b dx\; v^*(x)\, u(x) \; .$$

Notice that $(v,u)^* = (u,v)$ and, for any real function $f(x)$, we have $(v, fu) = (fv, u)$.

If a is a complex constant, we have:

$$(v, au) = a(v,u) \quad \text{but} \quad (av, u) = a^*(v,u)$$

We also have $\boxed{(u,u) \geq 0}$ for any function u

because we are integrating $|u(x)|^2$, which is pointwise nonnegative, over the interval $a \leq x \leq b$.

(u,u) can only be equal to zero if $u(x) = 0$ for all x in the interval.

Also, if f is a positive function in the interval, then $(u, fu) \geq 0$ with this equal to zero only if $u(x) = 0$.

Denote the eigenfunctions of the S-L problem as u_n:

$$\mathcal{L} u_n + \lambda_n w u_n = 0$$

Take the inner product with u_m:

① $$(u_m, \mathcal{L} u_n) + \lambda_n (u_m, w u_n) = 0$$

Recall $\mathcal{L}$ is Hermitian $(v, \mathcal{L}u) = (\mathcal{L}v, u)$

Take the complex conjugate of ①:

$$\underbrace{(u_m, \mathcal{L} u_n)^*}_{\underbrace{(\mathcal{L} u_n, u_m)}_{\substack{= (u_n, \mathcal{L} u_m) \\ \text{by Hermiticity of } \mathcal{L}}}} + \lambda_n^* \underbrace{(u_m, w\, u_n)^*}_{\underbrace{(w\, u_n, u_m)}_{\substack{= (u_n, w\, u_m) \\ \text{by reality of } w}}} = 0$$

$$(u_n, \mathcal{L} u_m) + \lambda_n^* (u_n, w\, u_m) = 0$$

Interchange the role of m, n:

$$② \quad (u_m, \mathcal{L} u_n) + \lambda_m^* (u_m, w\, u_n) = 0$$

$$① - ② \implies (\lambda_n - \lambda_m^*)\,(u_m, w\, u_n) = 0$$

For the case $m = n$, we know $(u_m, w\, u_m) \geq 0$ and is only zero for the trivial $u_m = 0$ case. Thus:

$$\boxed{\lambda_n = \lambda_n^*}$$

S-L eigenvalues are real.

Then ① - ② is $(\lambda_n - \lambda_m)\,(u_m, w\, u_n) = 0$.
For $\lambda_n \neq \lambda_m$, this implies that the u_m are orthogonal in the sense that

$$\boxed{(u_m, w\, u_n) = \int_a^b dx\, u_m^* \, w\, u_n = 0} \qquad \lambda_m \neq \lambda_n$$

S-L eigenfns are orthogonal wrt the weight fn w.

For a degeneracy s.t. u_1 and u_2 have common eigenvalue, just find linear combinations

$$U_1 = \alpha_1 u_1 + \beta_1 u_2 \quad \text{and} \quad U_2 = \alpha_2 u_1 + \beta_2 u_2$$

such that $(U_1, w U_2) = 0$. (operator version of Gram-Schmidt orthogonalization)

Just as in ordinary vector space, the eigenfunctions of the S-L operator form a complete basis set for the function space.

They span the space.

This is extremely important because it means any fn in the space can be expressed as an expansion in the eigenfns.

$$f(x) = \sum_{m=1}^{\infty} c_m u_m(x)$$ ← usually an infinite series (generalized Fourier series)

where

$$c_m = \int_a^b dx\; u_m^*(x)\, f(x)\, \underline{w(x)} = \langle u_m | f \rangle$$

"components" of the function f along the "unit vectors" $u_m(x)$ (generalized Fourier coefficients)

(If u's not normalized to 1,

$$c_m = \frac{\int_a^b dx\; u_m^* f w}{\int_a^b dx\; |u_m|^2 w} \quad)$$

If f is square integrable w.r.t. w over the range $a \leq x < b$, the series expansion must at least converge in the mean to f and norm wrt w is:

$$(f,f) = \langle f|f \rangle = \int_a^b dx\, |f|^2 w = \sum_{m=1}^{\infty} |c_m|^2 \quad *$$

$$= \sum_{m=1}^{\infty} c_m^* c_m = \sum_{m=1}^{\infty} \langle f|u_m \rangle \langle u_m|f \rangle$$

This is called a completeness relation.

If it holds for all "vectors" $f(x)$ in a function space defined in $a \leq x \leq b$, that is a necessary and sufficient condition for the set $\{u_m(x)\}$ to be complete w.r.t. that space.

Let's see how convergence in the mean arises

$\longrightarrow$ DETOUR COMING $\longleftarrow$

"Convergence in the mean" means we can find, for a function $f(x)$ and any arbitrarily small positive ε, a finite linear combination

$$S_N = \sum_{m=1}^{N} \alpha_m u_m(x)$$

such that $\int_a^b dx\, w(x)\, |f(x) - S_N|^2 < \varepsilon$

The best approximation for a given N is found with the coefficients $\alpha_m = c_m = \int_a^b dx\, w(x)\, u_m^*(x)\, f(x)$.

For these coefficients

$$\lim_{N\to\infty} \int_a^b dx\ w(x)\,|f(x)-S_N|^2 =$$

$$= \int_a^b dx\ w(x)\,|f(x)|^2 - \sum_{n=1}^{\infty} |c_n|^2 = 0$$

This does not imply pointwise convergence let alone uniform convergence of the series. However, if f is piecewise continuous with a square integrable first derivative over $a \le x \le b$, the eigenfn expansion converges absolutely and uniformly to f in all subintervals free of pts of discontinuity. At disc, converges to the arithmetic mean of the right and left hand limits of f. If no pts of discontinuity, and if f satisfies the b.c.'s imposed on $\{u_m\}$, the expansion converges uniformly throughout $a \le x \le b$.

DETOUR BEGINS: Use a variational principle to build the set of S-L eigenfunctions

$$\mathcal{L}u + \lambda w u = 0 \qquad \text{where} \qquad \mathcal{L}u = (pu')' + qu$$

$$(f, \mathcal{L}g) = \int_a^b dx\ f\,[(pg')' + qg]$$

$$= \int_a^b dx\ [-f'pg' + fqg] = -(f', pg') + (f, qg)$$

A functional is an operator whose argument is a function and for which the result of its operator on the argument is a number.

Define the functional $\Omega(f)$ for any fn $f(x)$ as:

$$\Omega(f) = (f', pf') - (f, qf)$$

$$= \int_a^b dx \left[p(f')^2 - q f^2 \right]$$

Assume f is real for simplicity.

Assume we only deal with fns satisfying S-L b.c.'s
"admissible functions"

Also define the norm squared of f (w.r.t. w):

$$N(f) = (f, wf) = \int_a^b dx \, w f^2$$

More generally, the bilinear functionals $\Omega(f,g)$ and $N(f,g)$ are:

$$\Omega(f,g) = (f', pg') - (f, qg) \qquad \Omega(f) = \Omega(f,f)$$

$$N(f,g) = (f, wg) \qquad N(f) = N(f,f)$$

As defined above, the functionals have the following properties:

$$\mathcal{N}(f,g) = \mathcal{N}(g,f)$$

$$\mathcal{N}(f+g) = \mathcal{N}(f) + \mathcal{N}(g) + 2\mathcal{N}(f,g)$$

$$\Omega(f,g) = \Omega(g,f)$$

$$\Omega(f+g) = \Omega(f) + \Omega(g) + 2\,\Omega(f,g)$$

$$\Omega(f,g) = -(f,\mathcal{L}g) = -(\mathcal{L}f,g)$$

Let's see how the eigenvalues + eigenfns of the S-L problem can be built one by one with the following minimisation problem. Use the fact, stated without proof, that there is always a lowest eigenvalue, which we'll call λ_1.

Look for a function f admissible that minimizes

$$R \equiv \frac{\Omega(f)}{\mathcal{N}(f)}$$

f will only be determined up to overall constant

Call this function $f = \psi_1$ and call the minimum value of R, λ_1.

$$\Omega(\psi_1) = \lambda_1 \mathcal{N}(\psi_1)$$

If we add any mixture of another (admissible) fn η

$$\Omega(\psi_1 + \epsilon\eta) \geq \lambda_1 N(\psi_1 + \epsilon\eta)$$

arbitrary constant

$$\cancel{\Omega(\psi_1)} + 2\epsilon\,\Omega(\psi_1,\eta) + \epsilon^2\Omega(\eta) \geq \cancel{\lambda_1 N(\psi_1)} + 2\epsilon\lambda_1 N(\psi_1,\eta) + \epsilon^2\lambda_1 N(\eta)$$

$$2\epsilon\left[\Omega(\psi_1,\eta) - \lambda_1 N(\psi_1,\eta)\right] + \epsilon^2\left[\Omega(\eta) - \lambda_1 N(\eta)\right] \geq 0$$

Now take ϵ arbitrarily small and ignore the term in ϵ^2.

$$2\epsilon\left[\Omega(\psi_1,\eta) - \lambda_1 N(\psi_1,\eta)\right] \geq 0$$

Unless $[\ \cdots\ ]$ vanishes, this can always be violated by appropriate sign choice of ϵ.

$$\therefore \quad \Omega(\psi_1,\eta) - \lambda_1 N(\psi_1,\eta) = 0$$

$$-(\mathcal{L}\psi_1, \eta) - \lambda_1 N(\psi_1,\eta) = 0$$

$$\int_a^b dx\,\left((p\psi_1')' + q\psi_1\right)\eta + \lambda_1\int_a^b dx\; w\,\psi_1\,\eta = 0$$

$$\int_a^b dx\,\underbrace{\left[(p\psi_1')' + q\psi_1 + \lambda_1 w\psi_1\right]}_{\therefore\ =0}\eta = 0$$

arbitrary admiss function

$$(p\psi_1')' + q\psi_1 + \lambda_1 w \psi_1 = 0$$

$$\boxed{\mathcal{L}\psi_1 + \lambda_1 w \psi_1 = 0}$$

Thus, $f = \psi_1$ which minimizes R is the eigenfunction of the S-L equation with the lowest eigenvalue λ_1.

Build the next eignfn by considering the same minimisation problem but with extra constraint that next fn f is orthogonal to ψ.

$$\mathcal{N}(\psi_1, f) = 0$$

We want the <u>next</u> smallest value of R. Call the function that satisfies this $f = \psi_2$ and call the value of R, λ_2.

$$\boxed{\Omega(\psi_2) = \lambda_2 \mathcal{N}(\psi_2) \qquad \mathcal{N}(\psi_1, \psi_2) = 0}$$

If we take $f = \psi_2 + \epsilon\eta$, with ϵ a constant and η an admiss. fn that's orthogonal to ψ_1, then we must have

$$\Omega(\psi_2 + \epsilon\eta) \geq \lambda_2 \mathcal{N}(\psi_2 + \epsilon\eta)$$

Rexpress η as $\eta = \xi - c\psi_1$ where $c = \dfrac{\mathcal{N}(\psi_1, \xi)}{\mathcal{N}(\psi_1)}$

This does satisfy

$$\mathcal{N}(\psi_1, \eta) = \mathcal{N}(\psi_1, \xi) - \frac{\mathcal{N}(\psi_1, \xi)\,\mathcal{N}(\psi_1)}{\mathcal{N}(\psi_1)} = 0$$

but ξ is unconstrained in the sense that it does not have to be orthogonal to ψ_1.

$$\therefore \quad \Omega(\psi_2 + \epsilon\xi - \epsilon c\psi_1) \geq \lambda_2 \mathcal{N}(\psi_2 + \epsilon\xi - \epsilon c\psi_1)$$

Expand LHS and RHS, use properties of $\mathcal{N}, \Omega$, and the fact that ψ_1 satisfies S-L eqn.

⋮

The inequality becomes

$$2\epsilon\left[\Omega(\psi_2, \xi) - \lambda_2 \mathcal{N}(\psi_2, \xi)\right] + \epsilon^2\left[\Omega(\xi) - \lambda_2 \mathcal{N}(\xi) + c^2(\lambda_2 - \lambda_1)\,\mathcal{N}(\psi_1)\right] \geq 0$$

Again, neglect ϵ^2 term and use fact that ξ is arbitrary fn.

$$\Longrightarrow \quad \mathcal{L}\psi_2 + \lambda_2 w \psi_2 = 0$$

So ψ_2 that minimizes R and is orthogonal to ψ_1, is ~~an~~ eigenfn of S-L with eigenvalue λ_2.

Iteratively build up the set of orthogonal eigenfns.

At ~~the~~ $(N+1)$th stage, look for f that minimizes $R = \Omega(f)/\mathcal{N}(f)$ subject to $\mathcal{N}(\psi_n, f) = f$ for $1 \le n \le N$. The resulting value of R will be λ_{N+1} and f will be ψ_{N+1}, the eigenfunction.

If $f(x)$ is an admissible function that is orthogonal to the first N eigenfns, $[\mathcal{N}(\psi_n, f) = 0, \quad 1 \le n \le N]$ then it satisfies

$$\boxed{\Omega(f) \ge \lambda_{N+1} \mathcal{N}(f)} \qquad *$$

Completeness of the set of S-L eigenfns:

Make partial sum; as an approx to fn f:

$$S_N(x) = \sum_{n=1}^{N} c_n u_n(x)$$

where $\quad c_n = (u_n, w f) = \mathcal{N}(u_n, f)$

Define $\quad f_N(x) \equiv f(x) - S_N(x)$

Want to show that, as $N \to \infty$, the discrepancy btwn our attempted expansion and the fn f tends to 0 in some sense.

Define $\quad a_N^2 \equiv \int_a^b dx\, w(x) \left(f_N(x)\right)^2 = (f_N, w f_N)$

If $a_N^2 \to 0$ as $N \to \infty$, good least squares fit $= \mathcal{N}(f_N)$

Assume lowest eigenvalue $\lambda_1 = 0$.

Can do this since $\lambda_1 > -\infty$. Shift $\mathcal{L}$ to $\tilde{\mathcal{L}} = \mathcal{L} + \lambda_1 w$ by defining $\tilde{Q} = Q + \lambda_1 w$. Then all eigenvalues are shifted to $\tilde{\lambda}_n = \lambda_n - \cancel{\lambda}_1 \geq 0$.

The eigenfns are not affected.

$$\lambda_1 = 0 \qquad \lambda_n \geq 0$$

Define $F_N(x) = \dfrac{f_N(x)}{a_N}$; this fn is normalized to $N(F_N) = 1$.

Consider $N(u_n, F_N)$:

$$N(u_n, F_N) = \frac{1}{a_N}\Big[\underbrace{N(u_n, f)}_{\equiv c_n} - \sum_{m=1}^{N} c_m \underbrace{N(u_n, u_m)}_{\delta_{nm}}$$

$$= \frac{1}{a_N}\left(c_n - \sum_{m=1}^{N} c_m \delta_{nm}\right)$$

For

$$1 \leq n \leq N: \quad N(u_n, F_N) = (u_n, w F_N) = 0$$

$$n \geq N+1: \quad N(u_n, F_N) = (u_n, w F_N) = \frac{c_n}{a_N}$$

So $F_N(x)$ is orthog to all u_n up to $n = N$. Thus it satisfies * on page 93.

$$\Omega(F_N) \geq \lambda_{N+1} \underbrace{N(F_N)}_{=1} \geq \lambda_{N+1}$$

But $a_N^2 \Omega(F_N) = \Omega(f_N) = \Omega(f - \sum_{n=1}^{N} c_n u_n)$

Using the properties of $\Omega(f,g)$, the RHS can be expanded and finally expressed as

$$a_N^2 \Omega(F_N) = \Omega(f) \underbrace{- \sum_{n=1}^{N} c_n^2 \lambda_n}$$

since $\lambda_n \geq 0$, this term is negative

Thus, combining $a_N^2 \leq \dfrac{\Omega(f)}{\Omega(F_N)}$ and

$\Omega(F_N) \geq \lambda_{N+1}$ from before, we have

$$\boxed{a_N^2 \leq \frac{\Omega(f)}{\lambda_{N+1}}}$$

$\Omega(f)$ is positive unless $f = c u_1$, in which case it's 0, since $\lambda_1 = 0$.

Thus, as $N \to \infty$ and the λ increase, $a_N \to 0$.

So taking $N \to \infty$ means we can accurately expand any admissible fn $f(x) = \sum_{n=1}^{\infty} c_n u_n(x)$.

<u>Thus the set of eigenfunctions $u_n(x)$ is complete.</u>

To summarize, we can expand as

$$f(x) = \sum_{n=1}^{\infty} c_n u_n(x) \quad \text{where} \quad c_n = \int_a^b dx' \, u_n^*(x') f(x') w(x')$$

Plugging the expression for the coefficients into $f(x)$:

$$f(x) = \int_a^b dx' \; \underbrace{w(x') \sum_{n=1}^{\infty} u_n^*(x') u_n(x)} \, f(x')$$

Recall defining property of Dirac δ function ("reasonable" f)

$$f(x) = \int_a^b dx' \; \underbrace{\delta(x-x')} \, f(x')$$

Thus $\sum_{n=1}^{\infty} w(x') u_n^*(x') u_n(x)$ behaves like $\delta(x-x')$

(Sometimes leave weight fn in integral $f(x) = \int dx' \, w(x') \delta(x-x') f(x')$
and put $\delta(x-x') = \sum u_n^*(x') u_n(x)$)

$$\boxed{\sum_{n=1}^{\infty} w(x') u_n^*(x') u_n(x) = \delta(x-x')}$$

Aside:

Ket notation

Having determined the necessary weight fn for a S-L problem

$$\langle v | u \rangle = \int_a^b dx \; w(x)\, v^*(x)\, u(x)$$

$|u\rangle$ represents the fn $u(x)$, which is a set of values (usually a continuum) taken on the interval $a \le x \le b$. This is like a vector $\vec{a} = (a_1, a_2, \ldots, a_n)$

We consider the value of the function at a particular point x, $u(x)$, to be like a component in an $|x\rangle$ basis

$$u(x) = \langle x | u \rangle$$

→ implies existence of set of basis vectors $|x\rangle$ such that $\langle x' | x \rangle = 0 \quad x \neq x'$.

We want an analogy to the ordinary vector space relations

$$|a\rangle = \sum_{j=1}^{N} a_j |e_j\rangle \qquad \text{with} \quad a_k = \langle e_k | a \rangle$$

such that $|u\rangle = \int_a^b dx$ $w(x)\, u(x)\, |x\rangle$

Then $\underset{u(x')}{\langle x' | u \rangle} = \int_a^b dx \; w(x)\, u(x) \langle x' | x \rangle$

behaves as $\delta(x - x')$

So $\langle x | x' \rangle = \dfrac{1}{w(x)}\, \delta(x - x')$

Our fn space can have a <u>basis</u> $|u_m\rangle$ representing the fns $u_m(x) = \langle x | u_m \rangle$

The fns satisfy the closure relation

$$\sum_m u_m^*(x')\, u_m(x) = \frac{1}{w(x')}\, \delta(x - x')$$

We can rewrite this:

$$\sum_m \langle u_m | x' \rangle \langle x | u_m \rangle \qquad = \qquad \langle x | x' \rangle$$

$$= \sum_m \langle x | u_m \rangle \langle u_m | x' \rangle$$

$$= \langle x | \underbrace{\sum_m |u_m\rangle\langle u_m|}_{=1} | x' \rangle$$

So a reexpression of closure $1 = \sum_m |u_m\rangle\langle u_m|$

In the $|x\rangle$ basis, $1 = \int_a^b dx\, w(x)\, |x\rangle\langle x|$.

Very handy to have lots of ways to write the unit operator.

See how simple example of differential equation fits into S-L framework.

$$\mathcal{L} = \frac{d^2}{dx^2} \qquad w(x) = 1$$

S.L. eqn. $\quad \mathcal{L}u = \dfrac{d^2u}{dx^2} = -\lambda u \equiv -k^2 u \quad$ (1-d wave eqn)

So $p(x) = 1$ and $q(x) = 0$. We have previously solved it:

$$u(x) = \begin{cases} A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x & \lambda > 0 \\ A\cosh\sqrt{-\lambda}\,x + B\sinh\sqrt{-\lambda}\,x & \lambda < 0 \\ Ax + B & \lambda = 0 \end{cases}$$

Boundary conditions determine choice of soln and λ. If b.c.'s are:

- $u(x+2\pi) = u(x)$ [periodic] then:

  $\lambda = m^2 > 0$ and $u_m(x) = A_m\cos mx + B_m \sin mx \qquad m = 0, 1, 2 \ldots$

- $u(0) = 0 = u(b)$ then:

  $\lambda = \dfrac{n^2\pi^2}{b^2} > 0$ and $u_n(x) = A_n \sin\dfrac{n\pi x}{b} \qquad n = 1, 2, 3, \ldots$

- $u'(0) = 0$ and $|u(\infty)| < \infty$ then:

  $\lambda = k^2 > 0$ and $u_k(x) = A_k\cos kx \qquad 0 \le k < \infty$

- $|u(\pm\infty)| < \infty$ then:

  $\lambda = k^2 > 0$ and $u_k(x) = A_k e^{ikx} \qquad -\infty < k < \infty$

Indeed these "Fourier fns" are the most familiar complete set of functions.

Consider the S-L eigenvalue equation

$$\mathcal{L}\, u_n(x) = \frac{d^2 u_n(x)}{dx^2} = -k_n^2\, u_n(x)$$

with solns Fourier fns $\propto e^{\pm i k_n x}$

Taking the S-L problem to consist of periodic boundary conditions on a finite interval:

$$u_n(x) = u_n(x+T) \qquad a \le x \le a+T$$

This implies $k_n T = 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$

The discrete set $\{k_n^2\} = \left\{\left(\frac{2\pi n}{T}\right)^2\right\}$ is called the spectrum of the operator $\mathcal{L} = \frac{d^2}{dx^2}$.

The normalized eigenfunctions are $u_n(x) = \frac{1}{\sqrt{T}}\, e^{i\left(\frac{2\pi n}{T}\right)x}$

$n = 0, \pm 1, \pm 2, \ldots$

They satisfy the orthogonality relation:

$$\int_a^{a+T} u_m^*(x)\, u_n(x)\, dx = \delta_{mn} \qquad (w=1)$$

An eigenfn expansion of a function $f(x)$ in terms of this basis is a Fourier series representation.

$$f(x) = \sum_{m=-\infty}^{+\infty} c_m \frac{1}{\sqrt{T}} \exp\left(\frac{i 2\pi m}{T} x\right) \qquad k_m = \frac{2\pi m}{T}$$

$$\equiv \frac{a_0}{2} + \sum_{m=1}^{\infty} \left[a_m \cos\left(\frac{2\pi m x}{T}\right) + b_m \sin\left(\frac{2\pi m x}{T}\right) \right]$$

where $c_m = \langle u_m | f \rangle = \frac{1}{\sqrt{T}} \int_a^{a+T} dx \, e^{-i\frac{2\pi m x}{T}} f(x)$

The completeness relation is

$$\int_a^{a+T} |f(x)|^2 \, dx = \sum_{m=-\infty}^{+\infty} |c_m|^2$$

This is an expression of Parseval's eqn. In terms of the Fourier sine + cosine series, it is

$$\frac{2}{T} \int_a^{a+T} |f(x)|^2 \, dx = \frac{a_0^2}{2} + \sum_{m=1}^{\infty} (a_m^2 + b_m^2)$$

If we replace the finite interval range by the entire real line and the periodicity over $a \to a+T$ by a boundedness condition $|u_n(\pm\infty)| < \infty$; $-\infty < x < \infty$ the eigenfns become $u_k(x) \propto e^{ikx}$ with $-\infty < k < \infty$. The eigenvalues $\lambda = k^2$ are continuous so we say the operator $\mathcal{L} = \frac{d^2}{dx^2}$ has a continuous spectrum.

In this case, the orthonormality relation is

$$\int_{-\infty}^{\infty} e^{ikx} e^{-ik'x} dx = (2\pi)\, \delta(k-k')$$

so the normalized eigenfns are $u_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$

In terms of this basis, an eigenfn expansion of a function $f(x)$ defined on $-\infty < x < \infty$ is

" $\sum_m \to \int dk$ "

$$f(x) = \int_{-\infty}^{+\infty} F(k)\, u_k(x)\, dk$$ where

$$F(k) = \langle u_k | f \rangle = \int_{-\infty}^{\infty} u_k^*(x')\, f(x')\, dx'$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx'\, e^{-ikx'} f(x')$$

$F(k)$ is the Fourier transform of $f(x)$ and

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk$ is a Fourier integral representation of $f(x)$. The completeness relation becomes $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(k)|^2 dk$. This is Plancherel's theorem. The closure relation is

$$\int_{-\infty}^{\infty} u_k^*(x')\, u_k(x)\, dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk = \delta(x-x')$$

This is the Fourier integral repn. of the δ function.

Problems with Cylindrical Symmetry

Bessel Functions & Applications

We saw the following eq in the radial coordinate arise from Helmholtz's and Laplace's eqns in cylindrical coordinates.

$$\frac{d}{d\rho}\Big(\underbrace{\rho}_{p(\rho)}\frac{dR}{d\rho}\Big)\underbrace{-\frac{m^2}{\rho}}_{q(\rho)}R+\underbrace{k^2}_{\lambda}\underbrace{\rho}_{w(\rho)}R=0$$

This is written here in the S-L form. It is standard to make change of variable to $x = k\rho$

$$\frac{d}{dx}\left(x\frac{dR}{dx}\right)-\frac{m^2}{x}R+xR=0$$

Bessel's eqn of order m

Seek solns of the Frobenius series type by expanding about the regular singular pt $x=0$.

$$R(x)=x^\alpha\sum_{n=0}^{\infty}a_n x^n$$

$$\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n x^{n+\alpha-1}+\sum_{n=0}^{\infty}(n+\alpha)a_n x^{n+\alpha-1}$$

$$-m^2\sum_{n=0}^{\infty}a_n x^{n+\alpha-1}+\sum_{n=0}^{\infty}a_n x^{n+\alpha+1}=0$$

The indicial equation for the power α arises from the term in $x^{\alpha-1}$:

$$[\alpha(\alpha-1)+\alpha-m^2]\,a_0=0 \longrightarrow \alpha=\pm m$$

Find recursion relation for coefficients a_k by looking at general power, eg., $x^{k+\alpha-1}$:

$$[(k+\alpha)(k+\alpha-1)+(k+\alpha)-m^2]\,a_k + a_{k-2} = 0$$

Using $\alpha=\pm m$, $\quad a_k = \dfrac{-a_{k-2}}{k(k\pm 2m)}$

Look at $\alpha=+m$ case first. $R(x) = x^m \sum_{k=0}^{\infty} a_k x^k$

with $a_k = \dfrac{-a_{k-2}}{k(k+2m)}$

Note the x^α power gives: $[(1+\alpha)\alpha+(1+\alpha)-m^2]\,a_1=0$

With $\alpha^2=m^2$, this is $(2\alpha+1)\,a_1=0$. So, unless $\alpha=\frac{-1}{2}$, $a_1=0$. Let's take $a_1=0$ generally. Then all a_k with k odd vanish.

$k=2n$:

$$a_{2n} = \frac{-a_{2n-2}}{2n(2n+2m)} = \left[\frac{-1}{(2)^2}\right]\frac{1}{n(n+m)}\cdot\left[\frac{-1}{(2)^2}\right]\frac{a_{2n-4}}{(n-1)(n+m-1)}$$

$$= \left[\frac{-1}{(2)^2}\right]^3 \frac{1}{\underset{\smile}{n}(n+m)\underset{\smile}{(n-1)}(n+m-1)\underset{\smile}{(n-2)}(n+m-2)}\, a_{2n-6} = \cdots = \frac{(-1)^n a_0}{(2^2)^n \underset{\smile}{n}!\,(n+m)\cdots(m+

It is conventional to define $a_0 \equiv \dfrac{1}{2^m\,\Gamma(m+1)}$.

$\Gamma(\nu)$ is the gamma function. There are several ways to define it.

$$\text{eg.} \quad \Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt \qquad \operatorname{Re} z > 0$$

No detail here. We'll just use the property $\Gamma(x) = (x-1)\,\Gamma(x-1)$ for $x > 1$.

For integral argument $\Gamma(m+1) = m!$

With a_0 above:

$$a_{2n} = \frac{(-1)^n}{2^{2n+m}\, n!\; \underbrace{(n+m)\underbrace{(n+m-1)\cdots\underbrace{(m+2)\underbrace{(m+1)\,\Gamma(m+1)}_{\Gamma(m+2)}}_{\Gamma(m+3)}}_{\cdots}}_{\Gamma(n+m+1)}}$$

So, conventionally $\quad a_{2n} = \dfrac{(-1)^n}{2^{2n+m}\, n!\, \Gamma(n+m+1)}$

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{m+2n}$$

Bessel function of order m.

Check out second solution with $\alpha = -m$:

$$J_{-m}(x) = \left(\frac{x}{2}\right)^{-m} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n-m+1)} \left(\frac{x}{2}\right)^{2n}$$

But we have to worry about the case $\alpha_1 - \alpha_2 = \text{int}$, i.e. $2m = $ even integer.

$$a_k = \frac{-\,a_{k-2}}{k(k-2m)}$$

problem: diverges when $k = 2m$

Each a_k is determined by a_{k-2} below it. Could we just start our series above a_{2m}?

$$J_{-m}(x) = \left(\frac{x}{2}\right)^{-m} \sum_{n=m}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n-m+1)} \left(\frac{x}{2}\right)^{2n}$$

Change dummy index to $s = n-m$

$$J_{-m}(x) = \left(\frac{x}{2}\right)^{-m} \sum_{s=0}^{\infty} \frac{(-1)^m (-1)^s \left(\frac{x}{2}\right)^{2s} \left(\frac{x}{2}\right)^{2m}}{(s+m)!\,\Gamma(s+1)}$$

$$= (-1)^m \left(\frac{x}{2}\right)^{m} \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s}}{s!\,\Gamma(s+m+1)} = (-1)^m J_m(x)$$

We have not generated a linearly independent soln.

OK, can take generalized Frob form

$J_m(x) \ln x + x^{-m} \sum_{n=0}^{\infty} a_n x^n$ <u>but</u> conventionally proceed as:

Define Neumann function and motivate it.

$$N_\nu(x) \equiv \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$$

This is well defined for ν noninteger. For ν integer, we can use l'Hôpital's rule:

for $\lim\limits_{x\to a} f(x) \to 0$ and $\lim\limits_{x\to a} g(x) \to 0$,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

$$N_m(x) = \lim_{\substack{\nu\to m \\ \text{integer}}} \left\{ \frac{\left[\left(\frac{dJ_\nu}{d\nu}\right)\cos\nu\pi - \pi\underbrace{\sin\nu\pi}_{0 \text{ for } \nu=m} J_\nu(x) - \left(\frac{dJ_{-\nu}}{d\nu}\right)\right]}{\pi\underbrace{\cos\nu\pi}_{\to(-1)^m}} \right\}$$

$$= \frac{1}{\pi}\left[\left(\frac{dJ_\nu}{d\nu}\right)\Bigg|_{\nu=m,\text{ integer}} - (-1)^m\left(\frac{dJ_{-\nu}}{d\nu}\right)\Bigg|_{\nu=m}\right] +$$

To fully explore, need detailed knowledge of Γ fn and its derivatives. Let's just check

1) a ln term arises in differentiating J_ν

2) N_m does satisfy Bessel's eq.

1) $$\frac{dJ_\nu}{d\nu} = \frac{d}{d\nu}\left[\left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n}\right]$$

$$= \frac{d}{d\nu}\left(\frac{x}{2}\right)^\nu \left[\sum \cdots\right] + \left(\frac{x}{2}\right)^\nu \frac{d}{d\nu}\left[\sum \cdots\right]$$

$$\frac{dy^\nu}{d\nu} = \frac{d}{d\nu}\left[e^{\nu \ln y}\right] = \ln y \, e^{\nu \ln y}$$

$$= \ln y \, e^{\ln(y^\nu)} = y^\nu \ln y$$

$$\therefore \quad \frac{d}{d\nu}\left(\frac{x}{2}\right)^\nu = \left(\frac{x}{2}\right)^\nu \ln\left(\frac{x}{2}\right)$$

Thus $\frac{dJ_\nu}{d\nu}$ has a term containing $J_\nu \ln x$ as expected

2) Take B's eq for J_m and for J_{-m} and diff wrt m

① $$\frac{d^2}{dx^2}\left(\frac{dJ_m}{dm}\right) + \frac{1}{x}\frac{d}{dx}\left(\frac{dJ_m}{dm}\right) + \left(1 - \frac{m^2}{x^2}\right)\frac{dJ_m}{dm} = \frac{2m}{x^2} J_m$$

② $$\frac{d^2}{dx^2}\left(\frac{dJ_{-m}}{dm}\right) + \frac{1}{x}\frac{d}{dx}\left(\frac{dJ_{-m}}{dm}\right) + \left(1 - \frac{m^2}{x^2}\right)\frac{dJ_{-m}}{dm} = \frac{2m}{x^2} J_{-m}$$

① $- (-1)^m$ ②: $$\frac{d^2}{dx^2}\left[\!\left[\frac{dJ_m}{dm} - (-1)^m \frac{dJ_{-m}}{dm}\right]\!\right] + \frac{1}{x}\frac{d}{dx}\left[\!\left[\;''\;\right]\!\right] + \left(1 - \frac{m^2}{x^2}\right)\left[\!\left[\;''\;\right]\!\right] =$$

*pg 107

$$\frac{2m}{x^2}\left[J_m - (-1)^m J_{-m}\right] = 0$$

$\left[\!\left[\;''\;\right]\!\right] = N_m$

N_m obeys B's eqn.

Behaviour at $x=0$ and asymptotically as $x \to \infty$:

$$J_m(x) \xrightarrow[x\to 0]{} \left(\frac{x}{2}\right)^m \left[\frac{(-1)^0}{0!\,\Gamma(m+1)}\right] = \frac{1}{\Gamma(m+1)}\left(\frac{x}{2}\right)^m$$

lowest order ($n=0$) term

All but J_0 vanish at $x=0$.

Neumann functions not well behaved at $x=0$.

$$N_0(x) \xrightarrow[x\to 0]{} \frac{2}{\pi}\left(\ln\frac{x}{2} + \gamma\right)$$

Euler-Mascheroni constant

$$N_m(x) \xrightarrow[x\to 0]{} -\frac{\Gamma(m)}{\pi}\left(\frac{x}{2}\right)^{-m}$$

OK for m noninteger too.

Asymptotically,

$$J_m(x) \xrightarrow[x\to\infty]{} \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

$$N_m(x) \xrightarrow[x\to\infty]{} \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

both oscillatory as $x \to \infty$

Define combinations analogous to $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$:

Hankel fns: $H_m^{(1)}(x) = J_m(x) + i\,N_m(x)$

$H_m^{(2)}(x) = J_m(x) - i\,N_m(x)$

$$H_m^{(1,2)} \xrightarrow[x\to\infty]{} \sqrt{\frac{2}{\pi x}}\exp\left[\pm i\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)\right]$$

As $x \to 0$, dominant behaviour is that of the Neumann components.

Relations between the functions

Use $$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{m+2n}$$

① Divide out the factor $\left(\frac{x}{2}\right)^m$ and differentiate the result with respect to x. This yields

$$\frac{d}{dx}\left[\frac{J_m(x)}{x^m}\right] = -\frac{J_{m+1}(x)}{x^m} \qquad m \geq 0$$

eg. $J_1(x) = -J_0'(x)$

② Multiply by x^m and differentiate wrt x. Yields

$$\frac{d}{dx}\left[x^m J_m(x)\right] = x^m J_{m-1}(x) \qquad m \geq 1$$

Combining these results:

$$J_{m+1} + J_{m-1} = \frac{2m}{x} J_m$$

$$J_{m+1} - J_{m-1} = -2\frac{dJ_m}{dx}$$

Generating Function for Bessel Functions

One way we found B's eq was from $\boxed{\begin{array}{c}\text{Helmholtz eq} \\ (\nabla^2+k^2)u=0\end{array}}$ in cylindrical coords, by separation of variables.

Ignoring z, reduce to a 2d problem (polar coords)

$$\rho^2 \frac{1}{R\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{dR}{\partial\rho}\right) + \underbrace{\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2}}_{\equiv -m^2} + k^2\rho^2 = 0$$

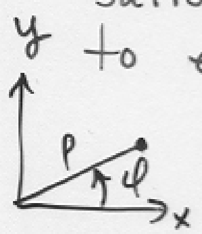
$$\Phi(\varphi) \sim e^{im\varphi} \qquad -\infty < m < \infty$$

$$R(k\rho) \sim J_m(k\rho)$$

Solns take form

$$u = R\Phi = \sum_k \sum_{m=-\infty}^{\infty} a_m J_m(k\rho) e^{im\varphi}$$

But we know that a plane wave, eg. $u = e^{iky}$, satisfies $(\nabla^2+k^2)u=0$. So we must be able to express this solution as expansion in Bessel fns



$y = \rho\sin\varphi$

$$e^{iky} = e^{ik\rho\sin\varphi} = \sum_{m'=-\infty}^{\infty} a_{m'} J_{m'}(k\rho) e^{im'\varphi} \qquad *$$

We can determine these coefficients using orthogonality of the functions $e^{im\varphi}$. We will find that all the coefficientes equal unity.

$$e^{ik\rho\sin\varphi} = \sum_{m=-\infty}^{\infty} e^{im\varphi} J_m(k\rho) \qquad \text{generating function}$$

Multiply each side of * by $e^{-im\varphi}$ and integrate over φ from $0 \longrightarrow 2\pi$:

$$\int_0^{2\pi} d\varphi\, e^{ik\rho\sin\varphi - im\varphi} = \underbrace{\sum_{m'=-\infty}^{\infty} a_{m'} J_{m'}(k\rho) \underbrace{\int_0^{2\pi} d\varphi\, e^{i(m'-m)\varphi}}_{=2\pi\delta_{mm'}}}_{=2\pi a_m J_m(k\rho)}$$

Evaluate $m=0$ and $m\neq 0$ cases separately.

$\boxed{m=0}$ a_0 is found simply using $\rho=0$ case ($J_0(0)=1$).

$$\int_0^{2\pi} d\varphi = 2\pi a_0 \quad \Rightarrow \quad \boxed{a_0 = 1}$$

$\boxed{m\neq 0}$ LHS: Expand $e^{ik\rho\sin\varphi}$ factor.

RHS: Use series expansion for $J_m(k\rho)$.

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-im\varphi}\left[1 + \sum_{n=1}^{\infty} \frac{(ik\rho\sin\varphi)^n}{n!}\right] d\varphi = a_m \left(\frac{k\rho}{2}\right)^m \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\,\Gamma(p+m+1)} \left(\frac{k\rho}{2}\right)^{2p}$$

integrates to 0 for $m\neq 0$.

Write $\sin\varphi = \frac{1}{2i}\left(e^{i\varphi} - e^{-i\varphi}\right)$.

Keep only the leading term in $(k\rho)$ on each side.

Evaluate for $m=1$ case:

$$\underbrace{\left(\frac{k\rho}{4\pi}\right)\int_0^{2\pi} e^{-i\varphi}\left(e^{i\varphi}-e^{-i\varphi}\right)d\varphi}_{\frac{k\rho}{2}} = \frac{a_1}{\Gamma(2)}\left(\frac{k\rho}{2}\right)$$

$\Gamma(2) = 1$

$$\Rightarrow \boxed{a_1 = 1}$$

At each successive order m, the first power $(k\rho \sin\varphi)^n$ that will contribute nonzero integral will be $n=m$. Generally,

$$\frac{1}{2\pi}\left(\frac{ik\rho}{2i}\right)^m \frac{1}{m!}\underbrace{\int_0^{2\pi} e^{-im\varphi}e^{im\varphi}\,d\varphi}_{2\pi} = \frac{a_m}{\underbrace{\Gamma(m+1)}_{=m!}}\left(\frac{k\rho}{2}\right)^m$$

$\therefore \boxed{a_m = 1}$ Thus, $e^{ik\rho\sin\varphi} = \sum_{m=-\infty}^{\infty} e^{im\varphi} J_m(k\rho)$

is a generating function for the Bessel fns.

The integral repn of the Bessel fn is:

$$J_m(k\rho) = \frac{1}{2\pi}\int^{2\pi} e^{ik\rho\sin\varphi - im\varphi}\,d\varphi$$

Write $\sin\varphi = \frac{1}{2i}\left(e^{i\varphi}-e^{-i\varphi}\right)$ and $e^{i\varphi} \equiv t$ to rewrite gen. fn:

$$e^{ik\rho\left(\frac{1}{2i}\right)(t-1/t)} = \exp\left[\frac{k\rho}{2}\left(t-1/t\right)\right] = \sum_{m=-\infty}^{\infty} t^m J_m(k\rho)$$

Orthogonality of the J_m

Since Bessel's equation is of the S-L form, the Bessel fns will satisfy an orthogonality relation providing they satisfy S-L boundary conditions on some interval.

Suppose the region in ρ we are interested in includes the origin and is finite $0 \le \rho \le a$. Since the N_m diverge at $\rho = 0$, only the J_m are allowed solns.

General superposition:

$$\sum_{m,k} \left[A J_m(k\rho) + \cancel{B}^{\,0} N_m(k\rho) \right] \cdot \left[C e^{im\varphi} + D e^{-im\varphi} \right] \cdot$$

or $\sin$ $\cos$

$$\cdot \left[D e^{\sqrt{k^2 - \lambda^2}\, z} + E e^{-\sqrt{k^2-\lambda^2}\, z} \right]$$

or $\sinh$ $\cosh$

One example of a S-L b.c. would be $\psi(\rho = 0)$ is finite and $\psi(\rho = a) = 0$. ← Dirichlet b.c. (Neumann if derivative vanishes)

$J_m(ka) = 0$ implies the eigenvalues k are such that ka = zeroes of J_m.

$$\boxed{k_{mn} = \frac{\alpha_{mn}}{a}}$$

α_{mn} is the nth zero of J_m.

Recall S-L situation:

$$\frac{d}{d\rho}\Big(\underset{\uparrow \atop p}{\rho}\frac{dJ_m(k_{mn}\rho)}{d\rho}\Big) - \underset{\uparrow \atop q}{\frac{m^2}{\rho}} J_m(k_{mn}\rho) + k_{mn}^2 \underset{\uparrow \atop w}{\rho} J_m(k_{mn}\rho) = 0$$

"Typo"

$$\mathcal{L}\, J_m(k_{mn}\rho) + k_{mn}^2\, w\, J_m(k_{mn}\rho) = 0$$

We find orthonormality relations by looking at:

$$(J_m(k_{mn'}\rho), \mathcal{L}\, J_m(k_{mn}\rho)) - (J_m(k_{mn}\rho), \mathcal{L}\, J_m(k_{mn'}\rho))$$

This yields:

$$\text{LHS} = \int_0^a d\rho\, J_m(k_{mn'}\rho)\Big[\frac{d}{d\rho}\Big(\rho\frac{dJ_m(k_{mn}\rho)}{d\rho}\Big) - \frac{m^2}{\rho}J_m(k_{mn}\rho)\Big]$$

$$-\int_0^a d\rho\, J_m(k_{mn}\rho)\Big[\frac{d}{d\rho}\Big(\rho\frac{dJ_m(k_{mn'}\rho)}{d\rho}\Big) - \frac{m^2}{\rho}J_m(k_{mn'}\rho)\Big]$$

$$= (k_{mn'}^2 - k_{mn}^2)\int_0^a d\rho\, \rho\, J_m(k_{mn'}\rho)\, J_m(k_{mn}\rho) = \text{RHS}$$

Integrating by parts, find LHS $= 0$. (using $J_m(k_{mn}a) = 0$)
Thus RHS must equal 0. For $n \neq n'$ ($k_{mn} \neq k_{mn'}$), we must have

$$n \neq n': \quad \int_0^a d\rho\, \rho\, J_m(k_{mn'}\rho)\, J_m(k_{mn}\rho) = 0$$

To normalize, we need $\int_0^a d\rho \, \rho \, J_m^2(k_{mn}\rho)$

Use a little trick. Go back to LHS = RHS but replace k_{mn} by k where ka is not a zero of J_m.

$$J_m(ka)\, a \left.\frac{d J_m(k_{mn}\rho)}{d\rho}\right|_{\rho=a} = (k^2 - k_{mn}^2) \int_0^a d\rho \, \rho \, J_m(k\rho) J_m(k_{mn}\rho)$$

Differentiate this expression wrt k. The $J_m(k\rho)$ are analytic fns of k

$$a^2 \left(\frac{d J_m(ka)}{d(ka)}\right) k_{mn} \left.\left(\frac{d J_m(k_{mn}\rho)}{d(k_{mn}\rho)}\right)\right|_{\rho=a} = 2k \int_0^a d\rho \, \rho \, J_m(k\rho) J_m(k_{mn}\rho) + (k^2 - k_{mn}^2) \frac{d}{dk} \int \cdots$$

Now set $k = k_{mn}$.

$$a^2 \left(J_m'(k_{mn}a)\right)^2 = 2 \int_0^a d\rho \, \rho \, J_m^2(k_{mn}\rho)$$

$$\int_0^a d\rho \, \rho \, J_m^2(k_{mn}\rho) = \frac{a^2}{2} \left(J_m'(k_{mn}a)\right)^2$$

prime indicates diff. wrt the argument $k\rho$ and ρ is set to a on RHS.

(Wyld does this normalization differently pg 141)

$$= \frac{a^2}{2} \left(J_{m+1}(k_{mn}a)\right)^2$$

(use pg 110 evaluated at $x = x_{mn}$.)

Modified Bessel Equation

Determination of eigenvalue (separation constant) m comes from angular dependence. If problem is independent of φ, $m=0$, implies radial solns. J_0, N_0.

But other separation constant (eigenvalue) k is hidden in the argument $x = k\rho$. This one comes into the z dependence as

$$\frac{d^2 Z}{dz^2} = (k^2 - \lambda^2) Z$$

constant in Helmholtz; if $\lambda = 0$, Laplace.

** → Laplace's eqn **

Ignoring λ, recall that we <u>chose</u> to write

$$\frac{d^2 Z}{dz^2} = +k^2 Z \quad \text{with solns } e^{\pm kz}$$

found $J_m, N_m (k\rho)$

- oscillatory fns with multiple zeroes

$\sinh z / \cosh z$

But constraints on k^2 come from b.c.'s. If, for instance, we had b.c. such that $Z(z)$ is periodic or such that $Z(0) = Z(b) = 0$, we would need to <u>choose</u>

$$\frac{d^2 Z}{dz^2} = -k^2 Z \quad \text{with oscillatory solns } e^{\pm ikz}$$

?(ρ)

$\sin z / \cos z$

↓ Sign of k^2 term in B's eqn changes.

$$\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) - \frac{m^2}{\rho}R - k^2\rho R = 0$$

was + before

Usual change of variable, yields $(x = k\rho)$

$$\frac{d}{dx}\left(x\frac{dR}{dx}\right) - xR - \frac{m^2}{x}R = 0$$ Modified Bessel eqn.

M.B. eq is just B. eq with $k^2 \to -k^2$ so the solutions are $J_m(ik\rho)$, $N_m(ik\rho)$. These fns are not always real. It is conventional to define the Modified Bessel fns

aka $\begin{cases}\text{Bessel fns of imaginary argument}\\ \text{Hyperbolic Bessel fns}\end{cases}$

such that they are real.

$$I_m(x) \equiv \frac{1}{i^m} J_m(ix)$$

$$= \frac{1}{i^m}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+m+1)}\left(\frac{ix}{2}\right)^{m+2n}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!\,\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{m+2n}$$ real

For integer m, I_{-m} is not linearly independent:

$$I_{-m}(x) = I_m(x)$$

So define our second solution using most common choice

$$K_m(x) \equiv \frac{\pi}{2} \frac{\left(I_{-m}(x) - I_m(x) \right)}{\sin \pi m}$$

Use same limiting procedure as for N_m to find

$$K_m(x) = \frac{(-1)^m}{2} \left[\frac{\partial I_{-m}(x)}{\partial m} - \frac{\partial I_m(x)}{\partial m} \right] \Bigg|_{m \text{ integer}}$$

$$= \frac{\pi}{2} i^{m+1} H_m^{(1)}(ix) \qquad \left(H_m^{(1)} = J_m + i N_m \right)$$

<u>Limiting forms</u>

$x \ll 1$: $\quad I_m(x) \longrightarrow \frac{1}{\Gamma(m+1)} \left(\frac{x}{2} \right)^m \qquad$ OK $\quad x = 0$

$$K_0(x) \longrightarrow -\left(\ln \frac{x}{2} + \gamma \right)$$

$$K_m(x) \longrightarrow \frac{\Gamma(m)}{2} \left(\frac{x}{2} \right)^{-m} \qquad m \neq 0$$

$\nearrow x = 0$

$x \gg 1, m$: $\quad I_m(x) \longrightarrow \frac{1}{\sqrt{2\pi x}} e^x \qquad \nearrow \quad x \to \infty$

$$K_m(x) \longrightarrow \sqrt{\frac{\pi}{2x}} e^{-x} \qquad \text{OK} \quad x \to \infty$$

I_0 has no zero. I_m, $m \neq 0$ vanishes at origin. Not orthogonal fns.
I's, K's more like exponentials.

So we see a correlation. (For Laplace's eqn)

z dependence: $\sinh(kz)/\cosh(kz)$ $\qquad$ $\sin kz/\cos kz$

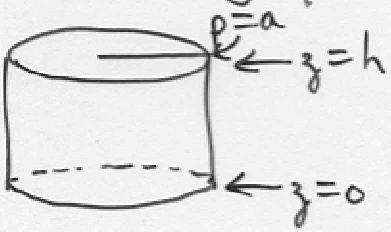
$$\begin{array}{ccc} \downarrow & \underline{\underline{vs}} & \downarrow \end{array}$$

ρ dependence: $J_m(k\rho)/N_m(k\rho)$ $\qquad$ $I_m\ /\ K_m$

① The $[(J/N);(\sinh/\cosh)]$ combination lets us satisfy zero boundary conditions in ρ. Eigenfns look like —

$$e^{\pm im\varphi}\cdot\left(A_{mn}\sinh(k_{mn}z)+B_{mn}\cosh(k_{mn}z)\right)\cdot\left(C_{mn}J_m(k_{mn}\rho)\right)$$

(assuming $\rho=0$ is included in range: dump N_m)



The fns $J_m(k_{mn}\rho)e^{\pm im\varphi}$ form a complete set on the $z=$ constant boundaries.

$$k_{mn}=\frac{\alpha_{mn}}{a} \leftarrow \text{zero of } J_m$$

② The $[(I/K);(\sin/\cos)]$ combination allows us to satisfy zero boundary conditions in z. Eigenfns —

$$\sin\left(\frac{n\pi z}{h}\right)\cdot\left[A_{mn}I_m\left(\frac{n\pi\rho}{h}\right)+\underbrace{B_{mn}}_{\text{dump if } \rho=0 \text{ included}}K_m\left(\frac{n\pi\rho}{h}\right)\right]e^{\pm im\varphi}$$

The fns $\sin\left(\frac{n\pi z}{h}\right)e^{\pm im\varphi}$ form a complete set on the $\rho=$ constant boundaries.

Solutions of Laplace's Equation in Cylindrical Coordinates

$$\nabla^2 \psi = 0 \qquad\qquad \psi = \sum_{m,n} c_{mn} \, \psi_{mn}$$

$$\psi_{mn} = \begin{Bmatrix} J_m(k_{mn}\rho) \\ N_m(k_{mn}\rho) \end{Bmatrix} \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix} \begin{Bmatrix} e^{-k_{mn} z} \\ e^{+k_{mn} z} \end{Bmatrix}$$

<u>or</u>

$$\psi_{mn} = \begin{Bmatrix} I_m(k_{\text{[redacted]}n}\rho) \\ K_m(k_{\text{[redacted]}n}\rho) \end{Bmatrix} \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix} \begin{Bmatrix} \cos k_{\text{[redacted]}n} z \\ \sin k_{\text{[redacted]}n} z \end{Bmatrix}$$

<u>or</u>

No z-dependence:

$$\psi_m = \begin{Bmatrix} \rho^{m} \\ \rho^{-m} \end{Bmatrix} \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix}$$

$\psi(\varphi+2\pi) = \psi(\varphi) \Rightarrow m = 0, 1, 2 \ldots$ integer

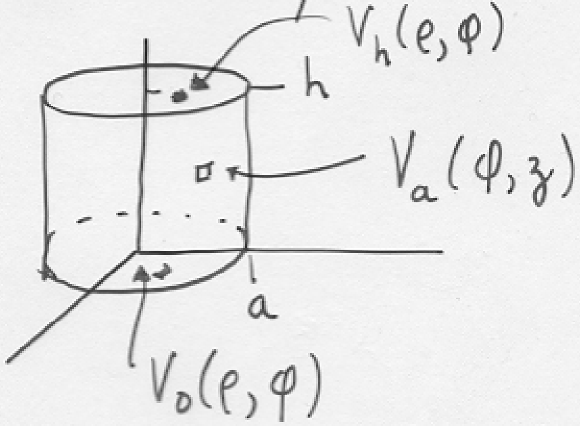
$\sum_{m=0} (Ae^{im\varphi} + Be^{-im\varphi})$

or $\sum_{m=-\infty}^{\infty} Ae^{im\varphi}$

Example Wyld Sec 4.7, 4.9

The volume of interest is a cylinder of radius $\rho = a$ and finite height from $z=0$ to $z=h$.

A value of the electromagnetic potential is specified on each surface. Solve $\boxed{\text{Laplace's eqn}}$ for the potential inside the cylinder.



$$\nabla^2 \psi = 0$$

$$\psi(\rho,\varphi,0) = V_0 = V_0(\rho,\varphi)$$

$$\psi(\rho,\varphi,h) = V_h = V_h(\rho,\varphi)$$

$$\psi(a,\varphi,z) = V_a = V_a(\varphi,z)$$

We can decompose the potential ψ into a sum of three functions $\psi = \psi_1 + \psi_2 + \psi_3$, where the ψ_i's satisfy —

$$\psi_1(\rho,\varphi,h) = V_h \qquad \psi_2(\rho,\varphi,h) = 0 = \psi_3(\rho,\varphi,h)$$

$$\psi_2(\rho,\varphi,0) = V_0 \qquad \psi_1(\rho,\varphi,0) = 0 = \psi_3(\rho,\varphi,0)$$

$$\psi_3(a,\varphi,z) = V_a \qquad \psi_1(a,\varphi,z) = 0 = \psi_2(a,\varphi,z)$$

This trick means we solve for a set of fns that has vanishing boundary conditions in either z or ρ.

First, because our region includes $\rho = 0$, we can only possibly find J's and I's.

$$\boxed{\text{no } N\text{'s or } K\text{'s}}$$

ψ_1 and ψ_2 vanish at $\rho = a$. So we expect

$$\psi_1, \psi_2 \sim J_m(\rho) \qquad \left[\begin{array}{l} \text{-finite at } \rho = 0 \\ \text{-zero at } \rho = a \end{array}\right]$$

correlation implies $\sinh kz$, $\cosh kz$

ψ_3 vanishes at both $z=0$ and $z=h$. So it must have oscillatory form in $z \rightarrow \sin kz$, $\cos kz$

$$\psi_3 \sim I_m(\rho) \longleftarrow \text{correlation implies}$$

① Look at ψ_1 first.
$$\psi_1(\rho, \varphi, h) = V_h(\rho, \varphi)$$
$$\psi_1(\rho, \varphi, 0) = 0$$
$$\psi_1(a, \varphi, z) = 0$$

$$\psi_1(\rho, \varphi, 0) = 0 \Longrightarrow (Ae^{kz} + Be^{-kz}) = 0 \text{ at } z = 0$$

$$B = -A : \quad \boxed{\sinh kz \text{ form}}$$

$$\psi_1 \propto \sinh kz$$

$\psi_1(a, \varphi, z) = 0$ implies $k_{mn} a = \alpha_{mn} \leftarrow$ nth zero of J_m.

$$\psi_1 \propto \sinh k_{mn} z \, J_m(k_{mn}\rho)\, e^{\pm im\varphi}$$

Denote $\displaystyle \psi_1 = \sum_{m=-\infty}^{\infty} e^{im\varphi} \sum_{n=1}^{\infty} C_{mn} J_{|m|}(k_{|m|n}\rho) \sinh(k_{|m|n} z)$

Determine the coefficients C_{mn} using the final b.c.

$\psi_1(\rho, \varphi, h) = V_h(\rho, \varphi)$ ← some given function

Multiply and integrate on each side with

$$\int_0^{2\pi} d\varphi\, e^{-im'\varphi} \int_0^a d\rho\, \rho\, J_{m'}(k_{|m'|n'}\rho)$$

$$V_h(\rho, \varphi) = \sum_{m=-\infty}^{\infty} e^{im\varphi} \sum_{n=1}^{\infty} C_{mn} J_{|m|}(k_{|m|n}\rho) \sinh(k_{|m|n} h)$$

This yields

$$C_{mn} = \frac{\int_0^{2\pi} d\varphi\, e^{-im\varphi} \int_0^a d\rho\, \rho\, J_{|m|}(k_{|m|n}\rho) V_h(\rho, \varphi)}{(2\pi)\left(\frac{a^2}{2}\right)\left(J'_{|m|}(k_{|m|n} a)\right)^2 \sinh(k_{|m|n} h)}$$

② Look at ψ_2 next.

$$\psi_2(\rho, \varphi, 0) = V_0(\rho, \varphi)$$

$$\psi_2(\rho, \varphi, h) = 0$$

$$\psi_1(a, \varphi, z) = 0$$

$\rho = 0$ includes $\rightarrow J_m(k\rho) \rightarrow e^{kz}, e^{-kz}$

$\psi_2(\rho, \varphi, h) = 0 \longrightarrow A e^{kh} + B e^{-kh} = 0 \quad \therefore B = -A e^{2kh}$

z dependence $\sim A\left[e^{kz} - e^{2kh} e^{-kz}\right] \sim A'\left[e^{k(z-h)} - e^{-k(z-h)}\right]$

$\psi_2 \sim \sinh(h - z)$

$\psi_2(a,\varphi,z)=0$ implies $k_{mn}\, a = \alpha_{mn}$ zeroes

$\psi_2 \propto \sinh k_{mn}(z-h)\; J_m(k_{mn}\rho)e^{\pm im\varphi}$

Denote as $\psi_2 = \sum_{m=-\infty}^{\infty} e^{im\varphi} \sum_{n=1}^{\infty} D_{mn}\, J_{|m|}(k_{|m|n}\rho) \sinh(k_{|m|n}(h-z))$

Determine the D_{mn} from $\psi_2(\rho,\varphi,0) = V_0(\rho,\varphi)$.

$$V_0(\rho,\varphi) = \sum_{m=-\infty}^{\infty} e^{im\varphi} \sum_{n=1}^{\infty} D_{mn}\, J_{|m|}(k_{|m|n}\rho) \sinh(k_{|m|n}h)$$

$$D_{mn} = \frac{\int_0^{2\pi} d\varphi\, e^{-im\varphi} \int_0^a d\rho\, \rho\, J_{|m|}(k_{|m|n}\rho)\, V_0(\rho,\varphi)}{(2\pi)\left(\frac{a^2}{2}\right)\left(J_{|m|}'(k_{|m|n}a)\right)^2 \sinh(k_{|m|n}h)}$$

③ Look at ψ_3. $\quad \psi_3(a,\varphi,z) = V_a(\varphi,z)$

$$\psi_3(\rho,\varphi,0) = 0$$

$$\psi_3(\rho,\varphi,h) = 0$$

$\psi_3(\rho,\varphi,0) = 0 \longrightarrow \psi_3 \sim \sin kz$

$\psi_3(\rho,\varphi,h) = 0 \longrightarrow kh = n\pi \quad \therefore k = n\pi/h$

$\psi_3 \propto \sin\left(\frac{n\pi z}{h}\right)$

Denote $\psi_z = \sum_{m=-\infty}^{\infty} e^{im\varphi} \sum_{n=1}^{\infty} E_{mn} \sin\left(\frac{n\pi z}{h}\right) I_{|m|}\left(\frac{n\pi\rho}{h}\right)$

Double Fourier series can be used to determine E_{mn} from $\psi_z(a,\varphi,z) = V_a(\varphi,z)$

$$E_{mn} = \frac{\int_0^{2\pi} d\varphi \, e^{-im\varphi} \int_0^h dz \, \sin\left(\frac{n\pi z}{h}\right) V_a(\varphi,z)}{(2\pi)\left(\frac{h}{2}\right) I_{|m|}\left(\frac{n\pi a}{h}\right)}$$

This was a general example for a region finite in z and ρ and including $\rho = 0$.

A remaining consideration is the circumstance that the region extends to $\rho = \infty$. This is similar to situation of going from finite to infinite range in, eg. z. Fourier series $\longrightarrow$ Fourier integral.

$-L \le z \le L \longrightarrow -\infty < z < \infty$

$\psi(z) \sim e^{in\pi z/L} \longrightarrow \psi(z) \sim e^{\pm ikz}$ $\quad$ n discrete $\hookrightarrow$ k continuous

$$\frac{1}{2L}\int_{-L}^{+L} \exp\left(\frac{in\pi z}{L}\right) \exp\left(-\frac{im\pi z}{L}\right) dz = \delta_{mn}$$

$$\hookrightarrow \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikz} e^{-ik'z} \, dz = \delta(k-k')$$

Extending the range of ρ from $0 \to \infty$ and imposing only finiteness conditions we go from

$$\int_0^a d\rho\, \rho\, J_m(k_{mn}\rho)\, J_m(k_{mn'}\rho) = \frac{a^2}{2}\left[J_m'(k_{mn}a)\right]^2 \delta_{nn'}$$

zeroes

to
$$\int_0^\infty d\rho\, \rho\, J_m(k\rho)\, J_m(k'\rho) = \frac{\delta(k-k')}{k} \qquad *$$

continuous

So a series expansion in Bessel fns

$$\sum_{n=1}^{\infty} c_{mn} J_m(k_{mn}\rho)$$

becomes an integral over a continuous parameter

$$\int_0^\infty dk\, A_m(k)\, J_m(k\rho)$$

The orthogonality relation * follows via the same method we used for the finite range in ρ (pg 114–116) but, to evaluate the functions at $\rho = \infty$, we use the asymptotic form for $J_m(k\rho)$ and the recursion formula

$$J_{m+1} - J_{m-1} = \frac{2\,dJ_m}{dx}.$$

From pg 115, LHS = RHS becomes here:

$$\rho\left[J_m(k'\rho)\frac{d J_m(k\rho)}{d\rho} - J_m(k\rho)\frac{d J_m(k'\rho)}{d\rho}\right]\Bigg|_0^\infty =$$

$$= \left[(k')^2 - k^2\right]\int_0^\infty d\rho\ \rho\, J_m(k\rho) J_m(k'\rho)$$

At the lower limit $\rho = 0$, the LHS vanishes.

Evaluate the first term on LHS at $\rho \to \infty$:

$$\lim_{\rho\to\infty} \frac{k\rho}{2} J_m(k'\rho)\left[J_{m-1}(k\rho) - J_{m+1}(k\rho)\right]$$

$$= \lim_{\rho\to\infty} \frac{k\rho}{2}\frac{2}{\pi\rho\sqrt{kk'}}\cos\left(k'\rho - \frac{m\pi}{2} - \frac{\pi}{4}\right)\left[\cos\left(k\rho - \frac{(m-1)\pi}{2} - \frac{\pi}{4}\right) - \cos\left(k\rho - \left(\frac{m+1}{2}\right)\pi - \frac{\pi}{4}\right)\right]$$

Using $\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$ and some further trig relations:

$$= \lim_{\rho\to\infty}\frac{1}{\sqrt{\pi}}\sqrt{\frac{k}{k'}}\left[(-1)^m \cos\left[(k+k')\rho\right] - \sin\left[(k-k')\rho\right]\right]$$

The second term on LHS is obtained by replacing $k \leftrightarrow k'$.

Together the terms yield:

$$\lim_{\rho\to\infty} \frac{1}{\pi}\left[\frac{(-1)^{m+1}}{\sqrt{kk'}} \frac{\cos(k+k')\rho}{k+k'} + \frac{1}{\sqrt{kk'}} \frac{\sin(k'-k)\rho}{k'-k} \right] \equiv \int_0^{\infty} J_m(k\rho) J_m(k'\rho) \cdot \rho d\rho$$

From $\displaystyle \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{+R} e^{ikr} dr = \delta(k)$, we

can determine that the sine term on LHS gives $\displaystyle \frac{1}{\sqrt{kk'}} \delta(k'-k) = \frac{\delta(k'-k)}{k}$.

The cosine term vanishes in the limit.

Finally, $\boxed{\displaystyle \int_0^{\infty} d\rho \, \rho \, J_m(k\rho) J_m(k'\rho) = \frac{1}{k} \delta(k'-k)}$

So, focusing just on ρ dependence, we can write a function $f(\rho)$ as an expansion in Bessel functions on the infinite range as

$$f(\rho) = \sum_n c_{mn} J_m(k_{mn}\rho)$$

$$f(\rho) \equiv \int_0^{\infty} dk \; k \, \underset{\uparrow \, c_{mn}}{F(k)} J_m(k\rho)$$

We can determine the coefficients $F(k)$ by multiplying each side by $\rho J_m(k'\rho)$, integrating ρ from $0 \to \infty$, and using the orthonormality.

$$\int_0^\infty d\rho\, \rho\, J_m(k'\rho)\, f(\rho) = \int_0^\infty dk\, k\, F(k) \underbrace{\int_0^\infty d\rho\, \rho\, J_m(k'\rho)\, J_m(k\rho)}_{= \frac{\delta(k-k')}{k}}$$

$$= F(k')$$

For $\displaystyle f(\rho) = \int_0^\infty dk\, k\, F(k)\, J_m(k\rho)$

we have

$$F(k) = \int_0^\infty d\rho\, f(\rho)\, J_m(k\rho)$$

This is the Fourier-Bessel integral transform.

(also called Hankel transform)

(analogous to Fourier integral transform pg 102

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx}\, F(k)$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx}\, f(x) \quad)$$

$$\varphi(\rho,\varphi,z) = \sum_{m=-\infty}^{+\infty} e^{im\varphi} \int_0^\infty dk\, A_m(k)\, f(k_z) J_{|m|}(k\rho)$$

Example with infinite range in ρ.

The electromagnetic potential on a plane at $z=0$ is given by some Known function of ρ. [for instance, $\varphi(\rho,\varphi,z=0) = V(\rho) = V_0 \frac{a}{\rho}\sin\left(\frac{\rho}{a}\right)$]

The potential obeys Laplace's eqn $\nabla^2\varphi = 0$.

Find the potential above the plane at $z>0$.

Angular dependence: none $\Rightarrow m=0$

z dependence: require $\varphi \to 0$ as $z \to \infty$.

$\Rightarrow f(kz) \sim e^{-kz}$

$$\varphi(\rho,\varphi,z) = \varphi(\rho,z) = \int_0^\infty dk\, A_0(k)\, e^{-kz} J_0(k\rho)$$

Use $z=0$ boundary condition:

$$V(\rho) = V_0 \frac{a}{\rho}\sin\left(\frac{\rho}{a}\right) = \int_0^\infty dk\, A_0(k)\, J_0(k\rho)$$

$\times \int_0^\infty d\rho\, \rho\, J_0(k'\rho)$ + use orthogonality to find $A_0(k)$

$$V_0 \int_0^\infty d\rho \, \rho \, J_0(k'\rho) \frac{a}{\rho} \sin\left(\frac{\rho}{a}\right) = \int_0^\infty dk \, A_0(k) \underbrace{\int_0^\infty d\rho \, \rho \, J_0(k'\rho) J_0(k\rho)}_{\frac{\delta(k-k')}{k}}$$

$$= \frac{A_0(k')}{k'}$$

Thus $$A_0(k) = k V_0 a \underbrace{\boxed{\int_0^\infty d\rho \, \sin\left(\frac{\rho}{a}\right) J_0(k\rho)}}_{\equiv I}$$

$$I = \begin{cases} 0 & \text{for } ka > 1 \\ \dfrac{a}{\sqrt{1-(ka)^2}} & \text{for } ka < 1 \end{cases}$$

$$\varphi(\rho, z) = V_0 a^2 \int_0^{1/a} dk \, \frac{k}{\sqrt{1-k^2a^2}} \, e^{-kz} J_0(k\rho)$$

$\leftarrow x = ka$

$$= V_0 \int_0^1 dx \, \frac{x}{\sqrt{1-x^2}} \, e^{-xz/a} J_0(x\rho/a)$$

(Fig 8.16 Lea)

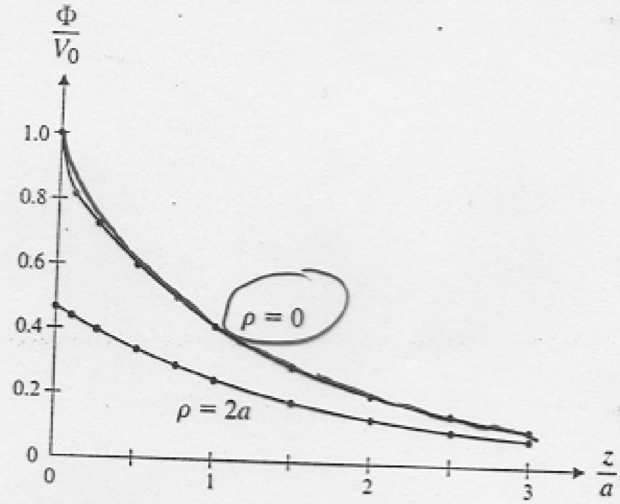
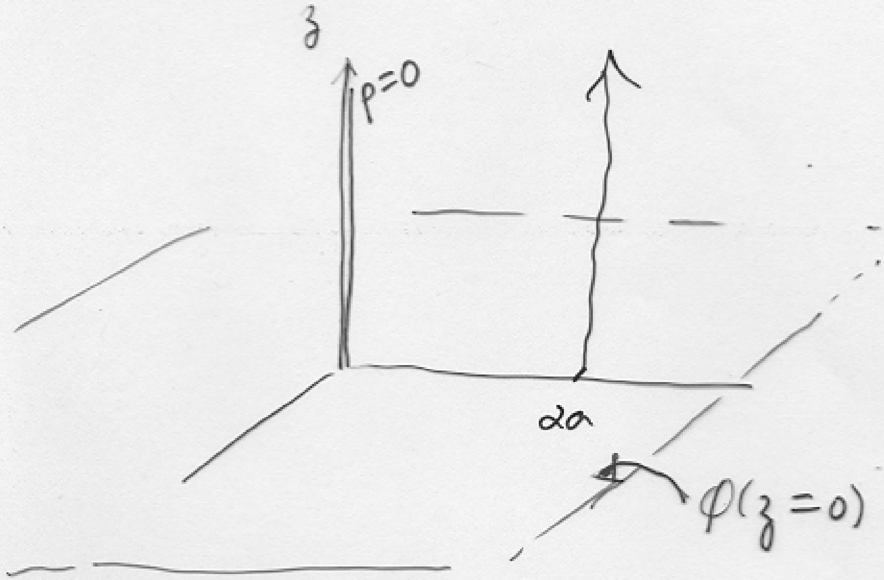


FIGURE 8.16. This plot of the solution to Example 8.6 shows $\Phi(\rho, z)$ on the z-axis ($\rho = 0$) and at $\rho = 2a$. The integration was done numerically.

(Lea)

$$\varphi(\rho, z) = V_0 \int_0^1 dx \frac{x}{\sqrt{1-x^2}} e^{-xz/a} J_0(x\rho/a)$$



$$\varphi(z=0) = V_0 \frac{a}{\rho} \sin(\rho/a)$$

Normal Mode Problems with Cylindrical Symmetry

Recall separation of time and spatial variables in diffusion equation and wave equation.

Spatial $\longrightarrow$ $\underbrace{\text{Helmholtz eqn.}}$ $\nabla^2 u(\vec{r}) + \lambda u(\vec{r}) = 0$

We have considered special case of

Laplace's equation ($\lambda = 0$) $\nabla^2 u(\vec{r}) = 0$
in cylindrical coordinates combined
with time independent, nonhomogeneous boundary condition.

Now treat diff + wave as:

→ time dependence separated

→ spatial dependence given by Helmholtz

reduce to combination of Laplace with time indpt nonhomogeneous b.c.

plus

Helmholtz with homogeneous boundary condition

To see method, consider diffusion eq first.

$$\nabla^2 \psi(\vec{r},t) = \frac{1}{K}\frac{\partial \psi(\vec{r},t)}{\partial t}$$

Solve this eq in some volume V of space subject to boundary conditions on its surface S and to some initial conditions.

$\psi(\vec{r},0)$ given in V

ψ given on S, independent of time

For instance: a finite size cylinder with interior temperature finite and initially given and with surface temperature vanishing at all times [redacted].

We know how to solve Laplace's eqn subject to time independent b.c.

$$\nabla^2 \psi'(\vec{r}) = 0 \qquad \psi'(\vec{r}) \text{ given on } S$$

Let say we find a ψ' that satisfies the time independent b.c. on S and obeys Laplace's eqn.

Try $$\psi(\vec{r},t) = \psi'(\vec{r}) + \underbrace{\psi''(\vec{r},t)}_{\text{What conditions on this?}}$$

$$\nabla^2 \psi(\vec{r},t) = \underbrace{\nabla^2 \psi'(\vec{r})}_{=0} + \nabla^2 \psi''(\vec{r},t)$$

$$= \frac{1}{K}\left[\underbrace{\frac{\partial}{\partial t}\psi'(\vec{r})}_{=0} + \frac{\partial}{\partial t}\psi''(\vec{r},t)\right]$$

$$\nabla^2 \psi''(\vec{r},t) = \frac{1}{K}\frac{\partial \psi''(\vec{r},t)}{\partial t}$$ ψ'' satisfies diff. eq.

Since $\psi'(\vec{r})$ satisfies the required time independent b.c. of $\psi(\vec{r},t)$ on S, it follows that $\psi''(\vec{r},t)$ must vanish on S.

So we have a pair of problems:

1) $\nabla^2 \psi'(\vec{r}) = 0$ $\psi'(\vec{r})$ given on S, indpt of time

2) $\nabla^2 \psi''(\vec{r},t) = \frac{1}{K}\frac{\partial \psi''(\vec{r},t)}{\partial t}$

$\psi''(\vec{r},t) = 0$ on S

$\psi''(\vec{r},0)$ given (initial condition)

Work on this second problem – with homogeneous boundary conditions – now.

$$\psi''(\vec{r},t) = T(t)\,u(\vec{r})$$

We separated this (pg 23) to:

$$\frac{dT}{dt} = -\lambda K T \longrightarrow \begin{cases} T_\lambda(t) \sim e^{-\lambda K t} & \lambda > 0 \\ T_\lambda(t) \sim e^{\lambda K t} & \lambda < 0 \\ T_\lambda(t) \sim \text{const} & \lambda = 0 \\ & \text{(Laplace)} \end{cases}$$

$$\nabla^2 u + \lambda u = 0$$

Take $\lambda = k^2 > 0$.

$$\psi''_n(\vec{r},t) = e^{-k_n^2 K t}\, u_n(\vec{r})$$

where
$$\boxed{\begin{array}{l} \nabla^2 u_n + k_n^2 u_n = 0 \\ \text{subject to } \quad u_n(\vec{r}) = 0 \quad \text{on } S \end{array}}$$

This forms a three dimensional S-L problem with orthogonal eigenfunctions u_n and eigenvalues k_n.

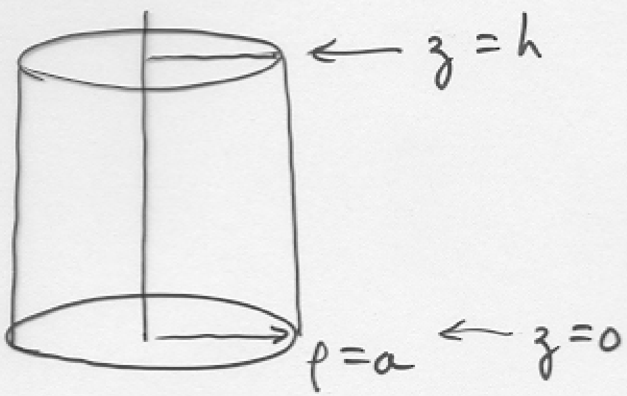
$$\psi''(\vec{r},t) = \sum_n c_n e^{-K k_n^2 t}\, u_n(\vec{r})$$

$$c_n = \frac{\int_V d^3x\, u_n(\vec{r})\, \psi''(\vec{r},0)}{\int d^3x\, |u_n(\vec{r})|^2}$$

given by initial condition

Heat conduction eqn.

Example: Heat conduction in a finite cylinder



Temperature $\psi(\vec{r}, t)$

$$\nabla^2 \psi = \frac{1}{\kappa} \frac{\partial \psi}{\partial t}$$

Say ψ vanishes on the surfaces of the cylinder. In this case, our b.c. is homogeneous so $\psi = \psi''$ — no need for solving Laplace's eqn separately.

$$\psi = T(t)\, u(\vec{r}) \qquad T(t) \sim e^{-\kappa k^2 t}$$

$$\nabla^2 u + k^2 u = 0 \qquad \leftarrow \text{Solved in cyl. coords pg 36- 38}$$

The form for u is:

$$u(\rho, \varphi, z) = \left[A e^{im\varphi} + B e^{-im\varphi} \right] \left[C e^{\sqrt{\alpha^2 - k^2}\, z} + D e^{-\sqrt{\alpha^2 - k^2}\, z} \right] \cdot \left[E J_m(\alpha\rho) + F N_m(\alpha\rho) \right]$$

αa must be a zero of J_m in order to make temperature vanish at $\rho = a$:

$$\alpha_{mn} = x_{mn} / a$$

x_{mn}: nth zero of $J_m(x)$

Must drop this to Keep temperature finite at $\rho = 0$.

$\boxed{\text{m integer for single-valuedness } u(\varphi + 2\pi) = u(\varphi)}$

To make temperatare vanish at both $z=0$ and $z=h$, the arguments $\sqrt{\alpha^2-k^2}\,z$ must give oscillating exponential.

$$\sqrt{\alpha^2-k^2} = i\beta$$

$$\left[Ce^{i\beta z}+De^{-i\beta z}\right] \quad \text{or} \quad \left[\underbrace{C\sin\beta z}+D\cos\beta z\right]$$

only this term survives in order to make temperature vanish at $z=0$.

For temperature to vanish at $z=h$, βh must be a multiple of π.

$$\beta h = l\pi \qquad l=1,2,3,\ldots$$

$$u_{mnl}(\rho,\varphi,z) = \left[A_m e^{im\varphi}+B_m e^{-im\varphi}\right]\sin\left(\frac{l\pi}{h}z\right)J_m(\alpha_{mn}\rho)$$

$$\psi(\vec{r},t) = \sum_{m,n,l} c_{mnl}u_{mnl}\,e^{-\kappa k^2 t}$$

determine coefficients by using initial condition.

$$k^2 = \alpha^2+\beta^2$$

$$= \alpha_{mn}^2 + \frac{l^2\pi^2}{h^2} \qquad l=1,2,3\ldots$$

$$= k_{mnl}^2$$

Similarly we can consider the wave equation.

$$\nabla^2 \psi(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \psi(\vec{r},t)}{\partial t^2} = 0$$

We will have
- a time independent boundary condition specifying ψ on S.
- an initial value of $\psi(\vec{r},0)$ in V
- $\left.\frac{\partial \psi(\vec{r},t)}{\partial t}\right|_{t=0}$ in V

two initial conditions since the time dependence is second order

Again divide the problem into two parts:

1) Find $\psi'(\vec{r})$ such that $\nabla^2 \psi'(\vec{r}) = 0$ and $\psi'(\vec{r})$ is subject to time independent b.c on S.

2) Find $\psi''(\vec{r},t)$ such that

$$\nabla^2 \psi''(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \psi''(\vec{r},t)}{\partial t^2} = 0$$

$\psi''(\vec{r},t) = 0$ on S

$\psi''(\vec{r},0)$ given in V

$\left.\frac{\partial \psi''(\vec{r},t)}{\partial t}\right|_{t=0}$ given in V

Same again $\psi''(\vec{r},t) = T(t)\,u(\vec{r})$

We separated the wave eqn on pg 23 to

$$\frac{d^2T}{dt^2} = -\lambda c^2 T \longrightarrow \begin{cases} T_\lambda(t) = A\cos\sqrt{\lambda}\,ct + B_\lambda \overset{\sin}{}\sqrt{\lambda}\,ct & \lambda > 0 \\ T_\lambda(t) = A' e^{\sqrt{-\lambda}\,ct} + B' e^{-\sqrt{-\lambda}\,ct} & \lambda < 0 \\ T_\lambda(t) = \tilde{A}ct + \tilde{B} & \lambda = 0 \text{ (Laplace)} \end{cases}$$

$$\nabla^2 u + \lambda u = 0$$

Typically a normal mode problem will have the $\lambda > 0$ oscillatory form for the time dependence, $\sqrt{\lambda} \equiv k_n$

$$\psi''(\vec{r},t) = \sum_n \left[A_n \cos k_n ct + B_n \sin k_n ct \right] u_n(\vec{r})$$

The initial value of $\psi''(\vec{r},0)$ determines this coefficient.

The initial value of $\left.\frac{\partial \psi(\vec{r},t)}{\partial t}\right|_{t=0}$ determines this coefficient.

$$A_n = \frac{\int_V d^3x\, u_n^*(\vec{r})\,\psi''(\vec{r},0)}{\int_V d^3x\, |u_n(\vec{r})|^2}$$

$$B_n = \frac{1}{k_n c} \frac{\int_V d^3x\, u_n^*(\vec{r}) \left.\frac{\partial \psi''(\vec{r},t)}{\partial t}\right|_{t=0}}{\int_V d^3x\, |u_n(\vec{r})|^2}$$

Wave eqn

Example: Vibrating Circular Drumhead

We had (pg 6) that the displacement $u(\vec{r},t)$ of a membrane of (mass/area $= \sigma$) subject to a tension T, with the weight negligible relative to the tension, obeyed the 2-dimensional wave equation.

$$\nabla^2 u(\vec{r},t) = \frac{1}{c^2}\frac{\partial^2 u(\vec{r},t)}{\partial t^2} \qquad c^2 = \frac{T}{\sigma}$$

The solution is a superposition of eigenfunctions

$$u(\vec{r},t) = \sum_n \left[A_n \cos k_n ct + B_n \sin k_n ct\right] u_n(\vec{r})$$

where the eigenfunctions satisfy the 2-d Helmholtz eqn

$$\nabla^2 u_n(\vec{r}) + k_n^2 u_n(\vec{r}) = 0$$

Assume a circular membrane:

We know the general form of solutions

$$u(\vec{r}) = u(\rho,\varphi) = \left[A J_m(k\rho) + \underbrace{B N_m(k\rho)}_{\to 0 \text{ for finiteness at } \rho=0.}\right]\left[C\cos m\varphi + D\sin m\varphi\right]$$

The standard time independent boundary condition is to have the membrane fixed at its radius $\rho = a$: $\boxed{u(a,\varphi)=0} \Longrightarrow k_{mn}a = x_{mn}$ nth zero of $J_m(x)$

non-vanishing at $t=0$

Thus

$$u(\vec{r},t) = \sum_{n,m} J_m(k_{mn}\rho)\Big[\left(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi\right)\cos k_{mn}ct$$
$$+ \left(C_{mn}\cos m\varphi + D_{mn}\sin m\varphi\right)\sin k_{mn}ct\Big]$$

$$k_{mn} = x_{mn}/a$$

derivative non-vanishing at $t=0$.

Initial conditions:

$$u(\vec{r},0) = \sum_{n,m} J_m(k_{mn}\rho)\left(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi\right)$$

$$\left.\frac{\partial u(\vec{r},t)}{\partial t}\right|_{t=0} = c\sum_{n,m} J_m(k_{mn}\rho)\left(C_{mn}\cos m\varphi + D_{mn}\sin m\varphi\right)k_{mn}$$

φ independent example $\longrightarrow$ $m=0$:

Initial conditions

$$u(\vec{r},0) = f(\rho) = -0.01a\, J_0\left(k_{01}\frac{\rho}{a}\right)$$

$k_{01} = x_{01}/a = 2.4048/a$

$$\left.\frac{\partial u(\vec{r},t)}{\partial t}\right|_{t=0} = g(\rho) = -0.1\,\frac{x_{02}}{a}\,c\, J_0(x_{02}\rho/a)$$

$k_{02} = x_{02}/a = 5.5201/a$

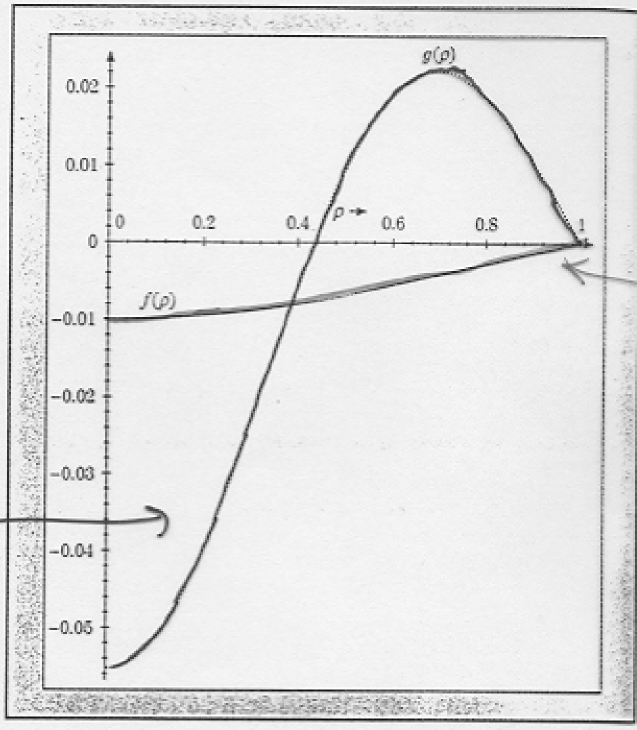
$$u(\vec{r},t) = \sum_n J_0(k_{0n}\rho)\left[A_n \underline{\cos k_{0n}ct} + B_n \sin k_{0n}ct\right]$$

$$\frac{\partial u(\vec{r},t)}{\partial t} = c\sum_n J_0(k_{0n}\rho)\left[-k_{0n}A_n\sin k_{0n}ct + k_{0n}B_n \underline{\cos k_{0n}ct}\right]$$

Figs from Weber + Arfken

Figure 12.4

Initial Central Bang on Drumhead



initial displacement

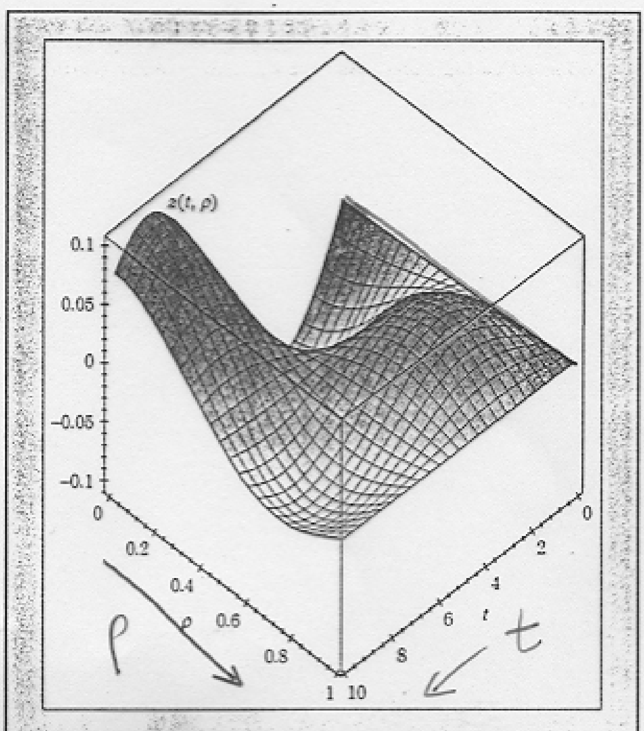
initial velocity

$$u(\vec{r},0) = f(\rho) = -0.01a\, J_0(x_{01}\rho/a) = \sum_n J_0(x_{0n}\rho/a)\, A_n$$

$$\left.\frac{\partial u(\vec{r},t)}{\partial t}\right|_{t=0} = g(\rho) = -0.1\,\frac{x_{02}c}{a}\, J_0(x_{02}\rho/a) = c\sum_n J_0(x_{0n}\rho/a)\, k_{0n} B_n$$

Figure 12.5

Drumhead Solution, $a = 1, c = 0.1$



$$u(\rho,t) = -0.01\,a\,J_0(2.4048\rho/a)\cos(2.4048ct/a) - 0.1\,J_0(5.5201\rho/a)\sin(5.5201ct/a)$$

Use orthogonality of Bessel fns to project out A_n, B_n.

$$\left[f(\rho) = -0.01\, a\, J_0(k_{01}\rho) = \sum_n J_0(k_{0n}\rho) A_n\right].$$

$$\times \int_0^a d\rho\, \rho\, J_0(k_{0n'}\rho)$$

→ pg 116

$$\int_0^a d\rho\, \rho\, J_m(k_{mn'}\rho) J_m(k_{mn}\rho) = \delta_{nn'} \frac{a^2}{2}\left[J_{m+1}(k_{mn}a)\right]^2$$

$$A_n = \frac{2}{a^2[J_1(k_{0n})]^2} \int_0^a d\rho\, \rho\, f(\rho)\, J_0(k_{0n}\rho)$$

$$= \frac{2}{a^2(J_1(k_{0n}))^2} \int_0^a d\rho\, \rho\, (-0.01a)\, J_0(k_{01}\rho)\, J_0(k_{0n}\rho)$$

$$= -0.01a\, \delta_{n1} \qquad A_1 = -0.01a$$

Similarly, $B_n = \dfrac{2}{a^2[J_1(k_{0n})]^2} \dfrac{1}{c k_{0n}} \displaystyle\int_0^a d\rho\, \rho\, g(\rho)\, J_0(k_{0n}\rho)$

$$= -0.1\, \delta_{n2}$$

$$u(\rho,t) = -0.01a\, J_0\left(2.4048\frac{\rho}{a}\right)\cos\Big(\underbrace{2.4048\frac{ct}{a}}_{=\,\omega_{01}}\Big) - 0.1\, J_0\left(5.5201\frac{\rho}{a}\right)\sin\Big(\underbrace{5.5201\frac{ct}{a}}_{\omega_{02}}\Big)$$

X_{mn} nth zero of J_m

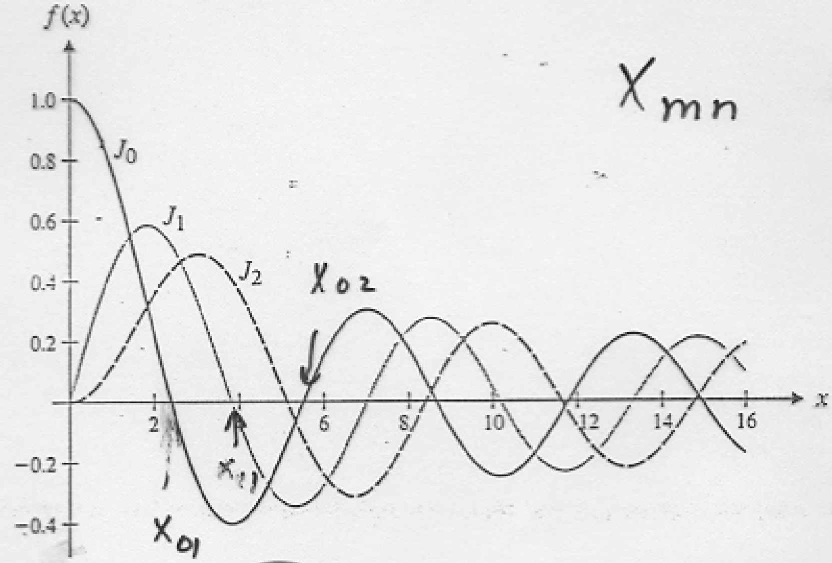


FIGURE 8.11. The first three Bessel functions. All the functions except $J_0(x)$ equal zero at $x = 0$, and all of them approach zero as $x \to \infty$. All of the J_m oscillate with decreasing amplitude.

Lea

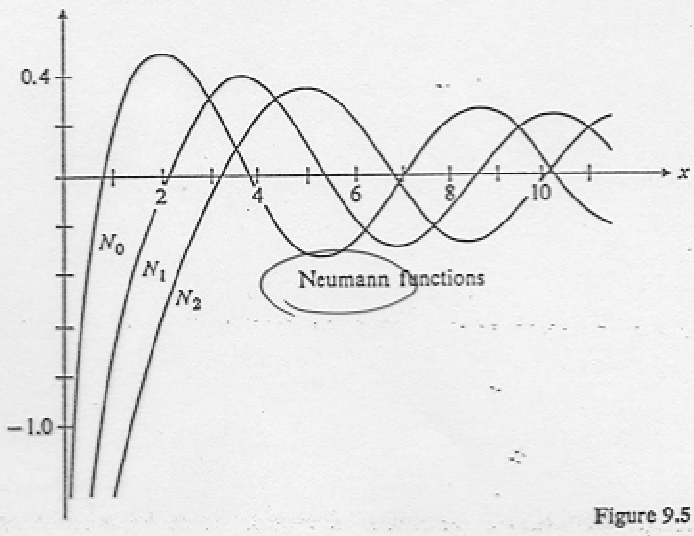


Figure 9.5 Butkov

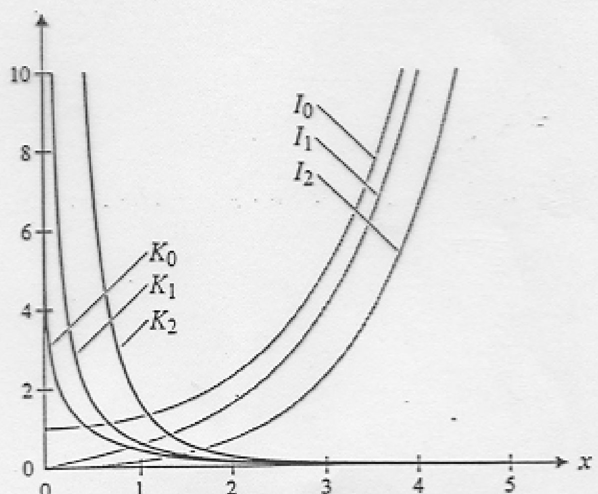


FIGURE 8.14. The first three modified Bessel functions. The functions $K_n(x)$ diverge at the origin, and the functions $I_n(x)$ diverge as $x \to \infty$.

Lea

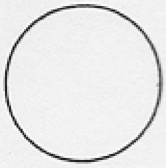
$r \to \rho$

$\theta \to \varphi$

Frequency

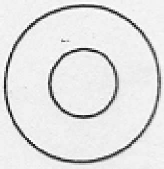
n = 1 m = 0 $J_0(k_{01}r)$

m



$\omega_{01} = \underbrace{2.405}_{X_{01}} \frac{c}{a}$

n = 2 m = 0 $J_0(k_{02}r)$



$\omega_{02} = \underbrace{5.520}_{X_{02}} \frac{c}{a}$

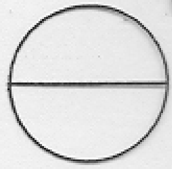
n = 3 m = 0 $J_0(k_{03}r)$

$\omega_{03} = \underbrace{8.654}_{X_{03}} \frac{c}{a}$

θ, φ

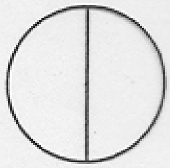
n = 1 m = 1 $J_1(k_{11}r)\sin\theta$

m



$\omega_{11} = \underbrace{3.832}_{X_{11}} \frac{c}{a}$

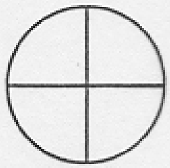
$J_1(k_{11}r)\cos\theta$



$\omega_{11} = 3.832 \frac{c}{a}$

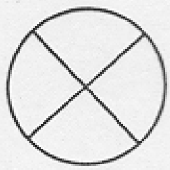
n = 1 m = 2 $J_2(k_{21}r)\sin 2\theta$

1



$\omega_{21} = \underbrace{5.136}_{X_{21}} \frac{c}{a}$

$J_2(k_{21}r)\cos 2\theta$



$\omega_{21} = 5.136 \frac{c}{a}$

Wyld

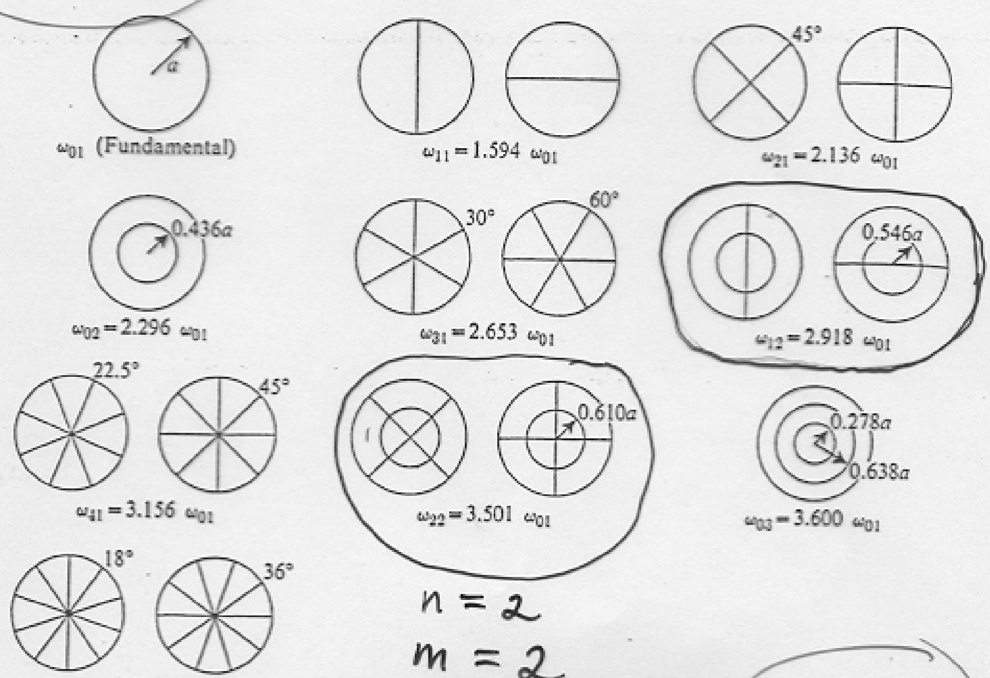
$e^{\pm i\omega_{mn}t}$

$\omega_{mn} = k_{mn}c$

$= X_{mn}\frac{c}{a}$

zero

Bat Kov



m = 1
n = 2

n = 2
m = 2

$u_{mn}(\rho, \varphi) \sim$

$J_m(k_{mn}\rho)\left[A_{mn}\cos(\varphi m)\right.$

$\left.+ B_{mn}\sin(m\varphi)\right]$

Figure 9.3