The Hitch Hikers Guide to the Quark Model

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Preface

Although quantum chromodynamics (QCD) is generally regarded as the theory that describes strong interaction physics we are as yet unable to make precise predictions for hadron properties. The reason for this is that QCD is a strongly interacting field theory and the techniques used in QED have limited applicability to QCD. In the *soft* QCD region relevant to hadron physics the situation is particularly difficult.

Unfortunately, high energy physicists have all but deserted the subject since they believe that since we know the Lagrangian all the rest is details. However there are a number of important reasons to study hadron physics. The first has already been stated; we still haven't solved QCD in the *soft* regime and we still don't understand hadrons from first principles. Until we do we cannot say that we understand QCD. The second reason is that there is a growing belief that the weak interactions become strong at the scale of electroweak symmetry breaking. If this turns out to be the case, low energy hadron dynamics will provide an important laboratory to study strong interactions which may have direct relevance to understanding electroweak symmetry breaking. The third reason is that knowledge of hadronic matrix elements is crucial for extracting electroweak parameters from experiment. Semileptonic B-decays and $B^0 - \bar{B}^0$ mixing are two examples. Finally, hadron physics is interesting in its own right with many interesting problems to understand. For example, at at present we don't even know what the correct degrees of freedom are for low energy hadron physics.

At Carleton a course is given to incoming particle physics graduate students on high energy physics phenomenology. It is split up into four parts of which I have given the section on hadron physics. Because I could not find any textbook that covered the subject in a way that

I was happy with I produced a set of notes which were the basis for this book. The material in this book is typically covered in 3 weeks of lectures consisting of 3 hour per week.

The course concentrates on the spectroscopy of hadrons. In other words the mass predictions of mesons and baryons and their transitions. As I said QCD has yet to be solved in this regime so we will turn to a model that has been very successful; the constituent quark model. When we are finished you will have the knowledge to calculate hadron properties which is turning out to be important to many topics in high energy physics such as extracting CKM matrix elements from B meson decay and studying CP violation at B factories.

Chapter 1

Introduction

In this book we will study the properties of hadrons, that is the properties of strongly interacting particles such as π's K's, ρ's, n, p, Λ, ... We now know that hadrons are made of quarks and antiquarks and the strong force is mediated by the exchange of gluons. The theory of quarks and gluons is Quantum Chromodynamics, a non-Abelian gauge theory. Strong interaction physics is much richer than hadron spectroscopy. It includes perturbative QCD and structure functions which are relevant to high energy processes, with high momentum transfer (Q^2). High Q^2 QCD is often referred to as *hard* QCD. The hadron physics I will restrict myself to is at low Q^2 which is sometimes referred to as *soft* QCD.

We find that the constituent quark model successfully describes much of hadron spectroscopy and as such is a useful tool for describing the properties of hadrons. However, one should not confuse the quark model with QCD. QCD is believed to give a much richer physical spectrum of states than that predicted by the quark model and includes objects called *glueballs*, *hermaphrodites*, and *multiquark states*. At present there is no irrefutable evidence for such exotics and it has yet to be shown from first principles that QCD does in fact predict them. The search for these new forms of hadronic matter has become a major preoccupation of hadron spectroscopists.

I plan on beginning with a bit of a historical introduction to hadron spectroscopy. That will also act as an overview of the subject. Next I will give some theoretical background to strong interaction physics and

Figure 1.1: A high Q^2 process.

some theoretical motivation to the quark model along with some other qualitative properties of hadrons.

Next we will look at the spectroscopy of heavy quarkonia including the spectroscopy, electromagnetic transitions and decays. The reasons that I begin with heavy quarkonia is that the basic premise of the quark model is on firmer foundation for the heavy quarkonia and the approach is taken over directly to light quark hadrons.

The next step is to extrapolate to light mesons where the non-relativistic approach is questionable, although even there it works better than it has any right to.

I start with the mesons since they are simplist to deal with than baryons so we will next turn to baryons but do not study them in the same detail as we studied mesons. Finally, in the last section, for completeness, I want to discuss quark model exotics; hybrids, glueballs, and multiquark states.

1.1 Historical Introduction

A good place to begin our brief history of hadron spectroscopy is 1947. At that time it appeared as if strong interaction physics was understood. The nucleus was made up of protons and neutrons and Yukawa's meson the π, was the mediater of the strong force which held together the nucleus.

The situation changed in December when Rochester and Butler discoverd a neutral particle which decayed into a π^+ and a π^- in a cosmic ray experiment. Since it looked like an upside down V in the cloud chamber photographs it was called the V^0[1] It had at least twice the mass of the pion. In 1949 Powell discovered the charged kaon in

$$K^+ \to \pi^+ + \pi^+ + \pi^- \tag{1.1}$$

which was originally called the τ^+. Because the kaon behavedin some respects like heavy pions they were included in the meson family.

Meanwhile, in 1950, another V particle was found by Anderson's group at Cal-Tech decaying to $p + \pi^-$. Since it decays to $p\pi^-$ it is a baryon and has mass greater than m_p. It is called the Λ. Over the next few years many more baryons were discovered, the Σ's, Ξ's, and the Δ's.

Some of these new particles had unexpected properties so they were referred to as strange paricles. They are produced copiously and therefore strongly but decayed relatively slowly and therefore weakly. This suggested that the production mechanism is different from their decay mechanism. They are produced by the strong force but decay by the weak force. Pais suggested that the strange particles are produced in pairs. Gell-Man and Nishijima assigned a new property, *strangeness*, which is conserved in any strong interaction but is not conserved in weak interactions. For example

$$\begin{aligned} \pi^- p &\to K^+\Sigma^- \\ &\to K^0\Sigma^0 \\ &\to K^0\Lambda \\ S &= +1 \; -1 \end{aligned} \tag{1.2}$$

and π and p have $S = 0$. You could never produce just one strange particle

$$\pi^- p \not\to \pi^+\Sigma^-. \tag{1.3}$$

On the other hand when these particles decay, strangeness is not conserved

$$\Lambda \to p\pi^-$$

[1]It was later known as the θ^0 and is now known as the K^0.

Figure 1.2: The Baryon Octet

$$\begin{aligned} \Sigma^+ &\rightarrow p\pi^0 \\ &\rightarrow n\pi^+. \end{aligned} \tag{1.4}$$

By 1960 there was a whole zoo or hadrons. They were divided into two general families; the baryons and the mesons.

1.2 The Eightfold Way

Gell-Man introduced the so called Eightfold way in 1961 as a way of arranging the baryons and mesons into geometrical patterns according to their charge and strangeness. The eight lightest baryons fit into the array: The eight lightest mesons formed a similar pattern, the (pseudo-scalar) meson octet. In addition there is the baryon decuplet: It was the prediction of the $Q = -1$, $S = -3$, Ω^- baryon and its subsequent discovery with the predicted properties that indicated that the Eightfold way was correct. Over the next ten years every new hadron found a place in one of the Eightfold way supermultiplets.

For the mesons, the antiparticles lie in the same supermultiplet as the corresponding particles. The Eightfold way classification of hadrons

Figure 1.3: The pseudo-scalar Meson Octet

Figure 1.4: The Baryon decuplet

Figure 1.5:

was the first step in understanding hadrons.

1.3 The Quark Model

So why do hadrons fit into the Eightfold way multiplets? In 1964 Gell-Man and Zweig independently proposed that all hadrons are composed of more elementary constituents, which Gell-Man called quarks (and Zweig called aces).

Gell-Mann viewed quarks as nothing more than mathematical devices while Zweig believed them to be real particles. For Zweig the origin of the quark model lay in the properties of the ϕ meson. In particular, the dominant decay mode was $\phi \to K\bar{K}$. In principle the ϕ should also decay to $\pi\pi$ and one would expect the $\pi\pi$ mode to dominate since the $K\bar{K}$ mode was just below threshold. In strong interaction physics one would expect anything that can occur to occur with maximum strength. So why was the $\pi\pi$ mode suppressed? History would repeat itself with the discovery of the J/ψ.

To understand this Zweig turned to the Sakata model with constituents Λ, n, p so that

$$\phi \sim \Lambda\bar{\Lambda} \tag{1.5}$$

and

$$\omega \sim (p\bar{p} + n\bar{n})/\sqrt{2}. \tag{1.6}$$

He interpreted meson decays as the separation of its constituents: Since

Figure 1.6: $\omega \to K^+K^-$

the ϕ was made of Λ and $\bar{\Lambda}$, constituents not in the ρ or π the decay $\phi \to \rho\pi$ could not occur.

Pursuing this, if one considers triplets of quarks

$$3 \times 3 \times 3 \tag{1.7}$$

one obtains the 8 and 10 representations of the eightfold way observed in nature.

One main results was that quarks had fractional charge. The coupling of hadrons to one another is determined by how the constituents could move from one hadron to another with the appropriate creation of pairs. For example, Zweig also looked at the possible quantum numbers that you can get from $q\bar{q}$ bound states. All his work is essentially our present picture of hadron spectroscopy!

The quarks come in three types (or flavours) The up (u) quark carries a charge of 2/3 and strangeness zero, the (d) down quark a charge of $-1/3$ and S=0, and the strange (s) quark charge $-1/3$ and S=-1. To each quark (q) there corresponds an antiquark ($\bar{q}$) with the opposite charge and strangeness.

The quark model asserts that

1. Every baryon is composed of 3 quarks (and every antibaryon is composed of 3 antiquarks).

2. Every meson is composed of a quark and an antiquark.

Figure 1.7: The fundamental quark triplet

Figure 1.8: The fundamental quark triplet

Table 1.1: The Baryron Decuplet

qqq	Q	S	Baryon
uuu	2	0	Δ^{++}
uud	1	0	Δ^{+}
udd	0	0	Δ^{0}
ddd	-1	0	Δ^{-}
uus	1	-1	Σ^{*+}
uds	0	-1	Σ^{*0}
dds	-1	-1	Σ^{*-}
uss	0	-2	Ξ^{*0}
dss	-1	-2	Ξ^{*-}
sss	-2	-3	Ω^{-}

Table 1.2: The Meson Nonet. The π^0, η, and η' are linear combinations of $u\bar{u}$, $d\bar{d}$ and $s\bar{s}$.

$u\bar{u}$	0	0	π^0
$u\bar{d}$	+1	0	π^+
$d\bar{u}$	-1	0	π^-
$d\bar{d}$	0	0	η
$u\bar{s}$	1	1	K^+
$d\bar{s}$	0	1	K^0
$s\bar{u}$	-1	-1	K^-
$s\bar{d}$	0	-1	$\bar{K}^0$
$s\bar{s}$ 0	0	η'	

1.4 Problems with the quark model

A problem with the quark model is that isolated quarks have never been seen. This led to widespread skepticism about the quark model and the existence of quarks. Supporters of the quark model introduced the notion of quark confinement in which; for reasons not understood, quarks are absolutely confined within baryons and mesons. This doesn't explain anything, it merely pushes the problem aside. However, the solution of the problem now appears to have been solved.

There was also a theoretical objection to the quark model. It appears to violate the Pauli exclusion principle that no two identical fermions can occupy the same state. For the case of the Δ^{++}, for example, the spin 3/2 Δ^{++} is composed of three identical u quarks;

$$|\Delta^{++}\rangle = |uuu\rangle|\uparrow\uparrow\uparrow\rangle \tag{1.8}$$

Since the Δ^{++} is a member of the supposed ground state baryon decuplet one would expect its wave function to be in a symmetric state. One way out was to suppose that the ground state wavefunction was not symmetric. However no reasonable wavefunction could be constructed. To get out of this dilema Greenberg proposed that the quarks had an additional quantum number called colour so that in addition to flavour quarks came in 3 colours. Therefore to properly anti-symmetrize the

Δ^{++} wavefunction the quarks would be an antisymmetric colour state and symmetric in spin, flavour, and spatial wavefunction. The antisymmetric colour wavefunction of a baryon is given by

$$\frac{1}{\sqrt{6}}\varepsilon_{ijk}u_iu_ju_k = \frac{1}{\sqrt{6}}[u_1u_2u_3-u_1u_3u_2+u_2u_3u_1-u_2u_1u_3+u_3u_1u_2-u_3u_2u_1] \tag{1.9}$$

and the total hadron wavefunction is given by

$$\psi = space \times spin \times flavour \times colour \tag{1.10}$$

This hypothesis seemed so ad hoc that many physicists did not take it very seriously and it was considered to be another problem of the quark model. Eventually it turned into the basis of QCD, the theory of the strong interactions. With the introduction of colour the idea of confinement was restated as only colourless objects occur in nature. This rule explains why particles can't be made of 2 quarks or 4 quarks and the only colourless hadrons are $q\bar{q}$, qqq, or maybe $qq\bar{q}\bar{q}$. The meson colour wavefunction is

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}(q_1\bar{q}_1 + q_2\bar{q}_2 + q_3\bar{q}_3) \tag{1.11}$$

1.5 The November Revolution (1974)

Although some physicists continued to work on the quark model and made progress, because of its ad-hoc assumptions it was not taken seriously by the physics community at large.

Then in 1974, a narrow hadron resonance was discovered independently by Ting at Brookhaven in $p\bar{p} \to \mu^+\mu^-$ and by Richter at SLAC in $e^+e^- \to hadrons$. Ting called it the J and Richter the ψ. It is known as the J/ψ. The J/ψ is an electrically neutral, extremely heavy meson with an exceedingly long lifetime $\tau \sim 10^{-20}$ sec. This should be compared to typical hadron lifetimes of the order of 10^{-23} sec so it is roughly 1000 times larger than comparable hadrons. This long lifetime indicated something fundamentally new and precipitated what is referred to as the November revolution.

The universally accepted interpretation is that the J/ψ is a bound state of a new found quark, the charm quark, and its antiquark so

Figure 1.9: The charmonium spectrum.

that $|J/\psi\rangle = |c\bar{c}\rangle$. (The idea of a fourth quark and its name was first introduced by Bjorken and Glashow in 1964.)

This interpretation had many implications including many new baryons and mesons. The charm quark is assigned a new quantum number +1 anc $\bar{c}$ with C=-1. In the J/ψ charm was hidden so to confirm this hypothesis it was important to produce a particle with naked (or bare) charm. The first charmed baryons $\Lambda_c^+ = udc$ and $\Sigma_c^{++} = ucs$ appeared in 1975, the first charmed mesons $D^0 = c\bar{u}$ and $D^+ = c\bar{d}$ in 1976, and the charm-strange meson $D_s = c\bar{s}$ in 1977.

Shortly thereafter another state, the ψ', was discovered at SLAC with mass 3685 GeV/c^2 and was interpreted as the first radial excitation of the J/ψ. The quark model predicted a rich spectroscopy of additional states, the η_c and χ_{c0}, χ_{c1}, χ_{c2}, and h_c. The $c\bar{c}$ energy levels are shown in fig??

In 1977 a new heavy mesons, the upsilon, was discovered and was understood to be the bound state of a fifth quark; the b for bottom or beauty so $\Upsilon = b\bar{b}$. The first beautiful baryon, $\Lambda = udb$ was discovered in 1981 and the first beautiful mesons $B^0 = b\bar{d}$ and $B^- = b\bar{u}$ were found in 1983. The bottomonium spectrum is shown in fig. ??

Figure 1.10: The bottomonium spectrum.

1.6 Quantum Chromodynamics

Part and parcel of this description of the heavy quarkonium system was the treatment of colour in the context of a gauge theory where a colour triplet of quarks interacts via a colour octet of massless gauge bosons called gluons in analogy to QED. A property of QCD is that quarks are confined respecting the ad-hoc ansatz of the quark model.

Calculational QCD has proven to be extremely difficult to solve in the soft, low Q^2 (large r) region. The problem is that the coupling constant of QCD, α_s, gets large for large separation and perturbation theory so useful in QED is no longer applicable. One approach to solving QCD is to evaluate the theory on a discrete space time lattice — Lattice QCD. However, it is likely to be some time before the thing will be considered to be solved. In the meantime the constituent quark model, with embelishments motivated to QCD, has proven to be an extremely useful and successful tool for study of hadron properties. Such potential models were pursued by Appelquist and Politzer.

Although colour was originally introduced in a rather ad hoc manner there is other evidence for 3 colours.

The first comes from e^+e^- annihilation into hadrons described by Comparing this to $e^+e^- \to \mu^+\mu^-$ and assuming that quarks hadronize in some manner independent of their production with unit probability

Figure 1.11: $e^+e^- \rightarrow hadrons$

the ration of these cross sections is

$$R = \frac{\sigma(e^+e^- \rightarrow hadrons)}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = \sum_{i=\substack{colours\\ flavours}} Q_i^2 \qquad (1.12)$$

where Q_i are the charges of all quarks which can be produced at a given energy.

High statistics data exists for centre of mass energies sufficient to produce u, d, s pairs but not heavier quarks or $\tau^+\tau^-$ pairs. We expect R in this energy range to be

$$R = N\{(\frac{2}{3})^2 + (\frac{1}{3})^2 + (\frac{1}{3})^2\} = \frac{2}{3}N \qquad (1.13)$$

where N is the number of colours. The data clearly favours $N = 3$. All e^+e^- data at higher energies are consistent with $N = 3$ once heavier quarks and $\tau^+\tau^-$ pairs are taken into account.

The second piece of evidenc comes from the decay of the neutral pion. A grossly simplified discussion follows.

The process $\pi^0 \rightarrow 2\gamma$ may be thought of as proceeding by means of an internal quark loop. The divergence of the axial-vector current $A_\mu^{(3)}$ carrying the π^0 quantum numbers is dominated by the π^0 pole. One obtains (using perturbation theory)

$$\Gamma(\pi^0 \rightarrow \gamma\gamma) = \frac{S^2 m_\pi^2}{8\pi f_\pi^2}(\frac{\alpha}{2\pi})^2 \qquad (1.14)$$

Figure 1.12: $e^+e^- \to hadrons$

Figure 1.13: $\pi \to \gamma\gamma$

where $f_\pi \simeq 130$ MeV and $S = N\{Q_u^2 - Q_d^2\} = N/3$. For $N = 1$ this predicts $\Gamma(\pi^0 \to \gamma\gamma) \simeq 8/9$ eV while for $N = 3$ this predicts $\Gamma(\pi^0 \to \gamma\gamma) \simeq 8$ eV. The experimental value is $\Gamma(\pi^0 \to \gamma\gamma) \simeq 7.95 \pm 0.55$ eV which favours N=3.

1.7 Spectroscopy Rules: Putting Quarks Together

To combine quarks and antiquarks to form $q\bar{q}$ mesons and qqq baryons start by coupling the quarks' spins together to obtain the total spin S and then couple to the orbital angular momentum, L, to obtain the total angular momentum J. Since quark spin and L need not be separately conserved quantum numbers, this prescription can sometimes lead to ambiguities.

1.7.1 Mesons

Let us begin with the $q\bar{q}$ system. The relative parity of q and $\bar{q}$ is negative. (Since P is represented in terms of Dirac matrices bye $\beta = \gamma_0$.) The parity of a spatial wavefunction with orbital angular momentum L is $(-1)^L$. Hence,

$$P(q\bar{q}, L) = (-1)^{L+1}. \tag{1.15}$$

A neutral $q\bar{q}$ system is also an eigenstate of the charge conjugation operator C with eigenvalues

$$C(q\bar{q}, L, S) = (-1)^{L+S} \tag{1.16}$$

It is useful to label all composite $q_i\bar{q}_j$ by this value of J^{PC} even when $i \neq j$. The composites formed in this way are summarized

The spectroscopic notation denotes ${}^{2S+1}L_J$ with S for L=0, P for L=1, D for L=2, and F, G, H, for L=3,4,5 etc. The restrictions of P and C restrict which states may mix with each other. P forbids mixing of even and odd values of L. The series $J^P = 0^+, 1^-, 2^+, 3^-$ ... is known as the natural parity states while the series $J^P = 0^-, 1^+, 2^-, 3^+$ ... is called unnatural.

Table 1.3: $q\bar{q}$ composites for $L \leq 2$

L	S	J^{PC}	Notation
0	0	0^{-+}	$^1S_0$
	1	1^{--}	$^3S_1$
1	0	1^{+-}	$^1P_1$
	1	0^{++}	$^3P_0$
		1^{++}	$^3P_1$
		2^{++}	$^3P_3$
2	0	2^{-+}	$^1D_2$
	1	1^{--}	$^3D_1$
		2^{--}	$^3D_2$
		3^{--}	$^3D_3$

Some combinations of J^{PC} cannot be made out of $q\bar{q}$. The state 0^{--} is forbidden as is the whole sequence 0^{++}, 1^{-+}, 2^{++} ... These are sometimes called "exotics of the second kind" and are a good signature for non-quark model states and more complicated configurations like hybrid ($q\bar{q} + g$) states, $qq\bar{q}\bar{q}$ multiquark states, and glueballs which have no constituent quark content at all.

The non-neutral $q\bar{q}$ composites fall into two classes with respect to mixing. Nonstrange with Q=0 $u\bar{d}$, $d\bar{u}$, are related to neutral ones $\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$ by isospin. Hence the selection rules appropriate to the neutral member are also appropriate to the charged one. We can define something called G-parity for the neutral member

$$G = C(-1)^I \tag{1.17}$$

All members of an isospin multiplet have the same G parity. Since $G(\pi) = -$ the G-parity counts the number of pions into which the state decays. If a state decays to both an even and an odd number of pions (as the J/ψ does) at least one class of decays must be violating isospin and therefore proceeds with essentially electromagnetic strength.

Strange, charmed, etc $q\bar{q}$ states are not C-eigenstates and only flavour symmetry presents the mixing of states with opposite C labels in Table 1. Thus the strange $^1P_1$ and $^3P_1$, $^1D_2$ and $^3D_2$, $^1F_3$ and

Table 1.4: Baryon quantum numbers and naming notation.

L	S	J^P	Notation	Other Possible States
0	1/2	$1/2^+$	$^2S_{1/2}$	$^4D_{1/2}$
	3/2	$3/2^+$	$^4S_{3/2}$	$^2D_{1/2}$, $^4D_{3/2}$
1	1/2	$1/2^-$	$^2P_{1/2}$	$^4P_{1/2}$
		$3/2^-$	$^2P_{3/2}$	$^4P_{3/2}$, $^4F_{3/2}$
	3/2	$1/2^-$	$^4P_{1/2}$	$^2P_{1/2}$
		$3/2^-$	$^4P_{3/2}$	$^2P_{3/2}$, $^4F_{3/2}$
		$5/2^-$	$^4P_{5/2}$	$^2F_{5/2}$, $^4F_{5/2}$
2	1/2	$3/2^+$	$^2D_{3/2}$	$^4S_{3/2}$, $^4D_{3/2}$
		$5/2^+$	$^2D_{5/2}$	$^4G_{5/2}$, $^4G_{5/2}$
	3/2	$1/2^+$	$^4D_{1/2}$	$^2D_{1/2}$
		$3/2^+$	$^4D_{3/2}$	$^4S_{3/2}$, $^2D_{3/2}$
		$5/2^+$	$^4D_{5/2}$	$^2D_{5/2}$, $^4G_{5/2}$
		$7/2^+$	$^4D_{7/2}$	$^2G_{7/2}$

$^3F_3$ states may mix with one another when the symmetry is broken.

1.7.2 Baryons

All quarks have +ve parity so for baryons we have

$$P = (-1)^L \tag{1.18}$$

For three spin 1/2 quarks possible total spins are S=1/2 and S=3/2. Combining orbital angular momentum and spin we obtain;

Because there are now three quarks with two angular momenta, three quark spins, and three quark flavours, there are a considerable number of possible baryon wavefunctions with many possible mixings. However, when two quarks in the baryon are identical or in the same isospin multiplet many restrictions follow. Nevertheless, the increasing number of possibilities makes the baryon wavefunctions considerably more complicated than the meson wavefunctions.

1.8 Naming Scheme for Hadrons

The Particle Data Group (PDG) has recently introduced a new hadron naming scheme to reduce the proliferation of mesons and to convey unambiguously the important quantum numbers of the particles they name. The quark model was used as a guide for the naming scheme without limiting it.

1.8.1 Neutral Flavour Mesons: S=C==B=T=0

Start with quantum numbers compatable with the quark model. Recall the spectroscopy notation of $q\bar{q}$ states:

$$
\begin{aligned}
&{}^{2S+1}L_J \\
P &= (-1)^{L+1} \\
C &= (-1)^{L+S}
\end{aligned}
$$

$^{2S+1}L_J$	$^1(L_{even})_J$	$^1(L_{odd})_J$	$^3(L_{even})_J$	$^3(L_{odd})_J$
PC	$-+$	$+-$	$--$	$++$
J^{PC}	0^{-+}	1^{+-}	1^{--}	0^{++}
	2^{-+}	3^{+-}	2^{--}	1^{++}
	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$u\bar{d},\ u\bar{u}-d\bar{d},\ d\bar{u}\ \ (I=1)$	π	b	ρ	a
$u\bar{u}+d\bar{d},\ s\bar{s}\ \ (I=0)$	$\eta,\ \eta'$	$h,\ h'$	$\omega,\ \phi$	$f,\ f'$
$c\bar{c}$	η_c	h_c	ψ	χ_c
$b\bar{b}$	η_b	h_b	Υ	χ_b
$t\bar{t}$	η_t	h_t	θ	χ_t

The entries in the table give the particle symbol. The spin J is added to the symbol as a subscript except for pseudoscalars and vector mesons. The mass is added in parenthesis fo any meson that decays strongly. Therefore the properties may be inferred unambiguously from the symbol. If the mass symbol cannot be assigned because the quantum numbers are unknown X is used. Sometimes it is not known whether a meson is mainly the isospin 0 mixture of $u\bar{u}+d\bar{d}$ or mainly $s\bar{s}$.

Gluonium states or other mesons that are not $q\bar{q}$ states are to be named just as $q\bar{q}$ states are named if the quantum numbers are not exotic. These states will likely be difficult to distinguish from $q\bar{q}$ states and will likely mix with them. An exotic meson with quantum numbers that a $q\bar{q}$ system cannot have;

$$J^{PC} = 0^{-+},\ 0^{++},\ 1^{-+},\ 2^{+-},\ 3^{-+} \tag{1.19}$$

will use the same symbol as would an ordinary meson that has all the same quantum numbers as the exotic meson except for C-parity. Then a hat is added to the symbol;

$$\begin{array}{ll} I = 1\ 0^{--} & \text{would be } \hat{\pi} \\ I = 0\ 1^{-+} & \text{would be } \hat{\omega} \end{array}$$

The results are:

	$I=1$	$I=0\ (ns)$	$s\bar{s}$	$c\bar{c}$	$b\bar{b}$
$^1S_0$	π	η	η'	η_c	η_b
$^3S_1$	ρ	ω	ϕ	J/ψ	Υ
$^1P_1$	b_1	h_1	h_1'	h_{c1}	h_{b1}
$^3P_0$	a_0	f_0	f_0'	χ_{c0}	χ_{b0}
$^3P_1$	a_1	f_1	f_1'	χ_{c1}	χ_{b1}
$^3P_2$	a_2	f_2	f_2'	χ_{c2}	χ_{b2}
$^1D_2$	π_2	η_2	η_2'	$\eta c2$	η_{b2}
$^3D_1$	ρ_1	ω_1	ϕ_1'	χ_{c0}	χ_{b0}
$^3D_2$	ρ_2	ω_2	ϕ_2'	χ_{c1}	χ_{b1}
$^3D_3$	ρ_3	ω_3	ϕ_3'	χ_{c2}	χ_{b2}

1.8.2 Charged Mesons

For Mesons with nonzero S, C, and T none of the states are eigenstates of charge conjugation and in each, one of the quarks must be heavier than the other. The rules are:

1. The main symbol is an upper case Roman letter indicating the heavier quark as follows

$$s \rightarrow \bar{K}\ c \rightarrow D\ b \rightarrow \bar{B}\ t \rightarrow T \tag{1.20}$$

This convention gives that the flavour carried by a charged meson has the same sign as its charge. Therefore K^+, D^+, B^+ have +ve strangeness, charm, and bottom.

2. If the other quark is not a u or d quark its identity is given by a subscript.

3. If the spin-parity is in the normal $J^P =)^+, 1^-, 2^+ \ldots$ a superscript is added.

4. The spin is added as a subscript except if the meson is a pseudoscalar or a vector.

Applying these rules results in the following meson names:

	$q\bar{s}$	$c\bar{q}$	$q\bar{b}$	$s\bar{b}$
$^1S_0$	K	D	B	B_s
$^3S_1$	K^*	D^*	B^*	B_s^*
$^1P_1$	K_1	D_1	B_1	B_{s1}
$^3P_0$	K_0^*	D_0^*	B_0^*	B_{s0}^*
$^3P_1$	K_1	D_1	B_1	B_{s1}
$^3P_2$	K_2^*	D_2^*	B_2^*	B_{s2}^*

1.8.3 Baryons

1. Baryons with three u and/or d quarks are named:

$$\begin{aligned} N\text{'s} \quad & I = 1/2 \\ \Delta\text{'s} \quad & I = 3/20 \end{aligned} \tag{1.21}$$

2. Baryons with 2 u and/or d quarks are:

$$\begin{aligned} \Lambda\text{'s} \quad & I = 0 \\ \Sigma\text{'s} \quad & I = 10 \end{aligned} \tag{1.22}$$

If the 3rd quark is a heavy quark (not s) its identity is given by a subscript; $\Lambda_c(2285)$, $\Sigma_c(2455)$, $\Lambda_b(5500)$.

3. Baryons with one u or d quark are Ξ's with $I = 1/2$. One or 2 subscripts are used if one or both of the remaining quarks are heavy; Ξ_c, Ξ_{cc}, Ξ_b.

4. Baryons with no u or d quarks are Ω's with $I = 0$. The subscripts indicate the heavy quark content.

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References

Chapter 2

Theoretical Background to Soft QCD

The property that distinguishes quarks from leptons is colour so it is natural to attempt to construct a theory of the strong interactions among quarks based on a colour gauge symmetry. The resulting theory is called quantum chromodynamics or QCD.

If QCD is the correct theory of the strong interactions it must describe an enormous range of phenomena, from the spectroscopy of light hadrons to deep inelastic scattering and jet formation at high energy colliders.

In this section I will look at the low Q^2 region of QCD sometimes referred to as *soft* QCD. I will look at

- Motivation for colour and $SU(3)$.
- The QCD Lagrangian.
- The strong coupling constant evolution.
- Qualitative pictures of *soft* QCD.
- Lattice QCD.
- The heavy quarkonium potential.

2.1 A Colour Gauge Theory

There is varied evidence that quarks are colour triplets. They are

- Resolution of the spin-statistics problem.
- Magnitude of the cross section for electron-positron annhilation into hadrons.
- The π^0 lifetime.
- It explains the branching ration for τ-decays.
- It is required for anomaly cancellation in the standard model

Attempts were made to formulate a dynamical theory based on colour symmetry. How did we arrive at $SU(3)$? Since quarks appear to be in colour triplets but hadrons are colour singlets possibilities are $SO(3)$, $SU(3)$, and $U(3)$.

In $SO(3)$ there is not distinction between colour and anticolour so there will be no distinction between quarks and antiquarks. The existence of $q\bar{q}$ mesons implies the existence of qq diquark states which would be fractionally charged. However fractionally charged diquarks are not observed so $SO(3)$ is not an appropriate choice.

In $U(3)$ colour you get a colour singlet gauge boson that occurs in $3 \otimes 3 = 1 \oplus 8$. It would mediate long range strong interactions between colour singlet hadrons and is therefore ruled out.

Therefore we are left with $SU(3)$.

The Lagrangian for $SU(3)$ colour is given by

$$L = i\bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{2}\mathrm{tr}(G_{\mu\nu}G^{\mu\nu}) \tag{2.1}$$

where the composite spinor for the colour triplet quarks is

$$\psi = \begin{matrix} q_{red} \\ q_{blue} \\ q_{green} \end{matrix} \quad)(2.2)$$

and the covariant derivative is

$$D_\mu = \partial_\mu + igB_\mu \tag{2.3}$$

Figure 2.1: The Quark-Gluon Vertex

where B_μ is a 3×3 matrix in colour space formed from the eight colour gauge fields b^i_μ and the generators $\lambda^i/2$ of $SU(3)$.

$$B_\mu = \frac{1}{2}\vec{\lambda} \cdot \vec{b}_\mu = \frac{1}{2}\lambda^l b^l_\mu \tag{2.4}$$

The gluon field strength term is

$$\begin{aligned} G_{\mu\nu} = \frac{1}{2}\vec{G}_{\mu\nu} \cdot \vec{\lambda} \qquad &= \tfrac{1}{2}G^l_{\mu\nu}\lambda^l \\ &= (ig)^{-1}[D_\nu, D_\mu] \\ &= \partial_\nu B_\mu - \partial_\mu B_\nu + ig[B_\nu, B_\mu] \end{aligned} \tag{2.5}$$

and

$$G^l_{\mu\nu} = \partial_\nu b^l_\mu - \partial_\mu b^l_\nu + gf^{ikl} b^j_\mu b^k_\nu \tag{2.6}$$

where f^{jkl} are the anti-symmetric structure functions of $SU(3)$. Knowing the QCD Lagrangian we can study the interactions between quarks.

The quark-gluon interaction term in the QCD Lagrangian is given by

$$L_{int} = -\frac{g}{2} b^a_\mu \bar{\psi}\gamma^\mu \lambda^a \psi \tag{2.7}$$

leading to the quark-gluon vertex given in fig.

So a quark with colour index $\alpha = R,\ B,\ G$ turns into a quark with colour index β and a gluon with Lorentz index μ and colour label $a = 1, 2, \ldots 8$. The one-gluon exchage force between quarks shown in fig. ? is proportional to

$$\sim \frac{g^2}{4} \sum_a \lambda^a_{\alpha\beta} \lambda^a_{\gamma\delta}. \tag{2.8}$$

To save writing define $\vec{T} = 1/2\vec{\lambda}$. In $SU(N)$ it is equivilant to average the square of any single generator over the representation or to perform

Figure 2.2: The Quark-Gluon Vertex

Table 2.1: Value of some colour Casimir operators in some representations.

representation	$\langle \vec{T}^2 \rangle$
1	0
3 or 3^*	4/3
6 or 6^*	10/3

the sum over all generations. The former is simpler and it is particularly convenient to choose I_3, the 3rd component of isospin in the flavour analogy.

The expectation value in a representation of dimension d is

$$\langle \vec{T}^2 \rangle_d = (N^2 - 1) \sum_{representation} I_3^2/d \tag{2.9}$$

where $N^2 - 1$ is the number of generators of $SU(N)$. To evaluate $\langle T^{(1)} \cdot T^{(2)}$ use the relation

$$\langle T^{(1)} \cdot T^{(2)} \rangle = \frac{1}{2}[\langle T^2 \rangle - \langle {T^{(1)}}^2 \rangle - \langle {T^{(2)}}^2 \rangle] \tag{2.10}$$

We can use this expression to evaluate the colour expectation value for various quark configurations. Some examples are given in the table ??

For $q\bar{q}$ system the one-gluon-exchange is attractive for the colour single but repulsive for the colour octet. Similary, for diquark systems, the colour triple is attractive but the sextet is repulsive.

For 3 (or more) body systems we assume that the interaction is the sum of 2-body forces so that

$$\sum_{i<j} \langle T^{(i)} \cdot T^{(j)} \rangle = \frac{1}{2}[\langle T^2 \rangle - \sum_i \langle T^{(i)2} \rangle]. \tag{2.11}$$

Table 2.2: The interaction energies for a few quark systems.

configuration	$\langle \sum_{i<j} T^{(1)} \cdot T^{(2)} \rangle$
$\langle q\bar{q} \rangle_1$	-4/3 (attractive)
$\langle q\bar{q} \rangle_8$	1/6 (repulsive)
$\langle q\bar{q} \rangle_{\bar{3}}$	-2/3 (attractive)
$\langle qq \rangle_6$	1/3 (repulsive)
$\langle qqq \rangle_1$	-2 (attractive)

These simple observations do not show the richness of QCD and the difficulty in solving it. We have neglected multigluon exchange and the trilinear gluon coupling.

2.2 Charge Renormalization in QED

2.3 The Running Coupling Constant in QCD

2.4 Qualitative Models of QCD

2.5 Lattice QCD

Chapter 3

The Spectroscopy of Heavy Quarkonia

3.1 The Spin-Independent Potential

In the previous chapter I gave qualitative arguments why the spin-independent potential is a linear plus Coulomb potential

$$V(r) = -\frac{4}{3}\frac{\alpha_s(r)}{r} + br \tag{3.1}$$

with $b \simeq 0.18$ GeV^2. We also saw how this potential is consistent with results from lattice QCD. However, historically this form was arrived at through trial and error (although Appelquist and Politzer got i right in a early paper $\sim$ 1975!). Emperically, the Schrodinger equation was solved for a given potential which was modified until agreement was achieved between theory and experiment.

We have

$$M = m_1 + m_2 + E_{nl} \tag{3.2}$$

where

$$\left[\frac{p^2}{2\mu} + V(r)\right]\psi = E_{nl}\psi \qquad \text{and} \qquad \left(\mu = \frac{m_1 m_2}{m_1 + m_2}\right) \tag{3.3}$$

This gives

$$\left[-\frac{\hbar}{2\mu}\nabla^2 + V(r)\right]\psi = E_{nl}\psi \tag{3.4}$$

with

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{3.5}$$

Separating variables

$$\psi(r,\theta,\phi) = R(r)Y_{lm}(\theta\phi) \tag{3.6}$$

results in

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \left[\frac{l(l+1)}{r^2} + \frac{2m}{\hbar^2}V(r) - E\right]R \tag{3.7}$$

substituting $U(r) = rR(r)$

$$-\frac{\hbar^2}{2\mu}\frac{d^2U}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]U = EU \tag{3.8}$$

with boundary conditions $U(0) = 0$ and $U'(0) = R(0)$. To see the phenomenological motivation for the linear plus Coulomb potential consider the harmonic oscillator and Coulomb energy levels and then we see that the $c\bar{c}$ lies somewhere between the two. Therefore the linear plus Coulomb potential is a reasonable interpolation between the two. We could see this in another way by starting with the energy levels and wavefunctions of either and then treating the other as a perturbation. In the following figure we show the $c\bar{c}$ and $b\bar{b}$ spectra. Phenomenologically we find

$$\Delta(\psi(3685) - \psi(3097)) \simeq \Delta(\Upsilon(10023) - \Upsilon(9460)) \tag{3.9}$$

Two potentials which reproduce these splittings are:

$$\begin{cases} V(r) = \lambda r^\nu & r \simeq 0.1 \\ V(r) = c\ln(r/r_0) & c = 0.73 \end{cases} \tag{3.10}$$

We could also fit these splittings with the linear plus Coulomb potential for suitable values of α_s and b. We could also use the postion of the P-wave states as a criteria for a suitable potential. The spin-averaged ^3P_J states gives

$$\bar{M} = (5M(^3P_2) + 3M(^3P_1) + M(^3P_0)/9 \tag{3.11}$$

Figure 3.1: Comparison of the energy levels of the Harmonic oscillator and Coulomb potentials to the $c\bar{c}$ spectrum.

Figure 3.2: The charmonium and bottomonium spectra

For $c\bar{c}$ $\bar{M} = 3522$ MeV

$$\frac{M(2S) - M(1P)}{M(2S) - M(1S)} = \begin{cases} 1/2 & \text{H.O.} \qquad (\nu = 2) \\ 1/4 & \text{for} \nu = 0 \\ 0 & \text{Coulomb} \qquad (\nu = -1) \end{cases} \tag{3.12}$$

This gives from $c\bar{c}$ $\nu \simeq 0.15$.

3.2 Spin Dependent Potentials

In general one would expect spin-dependent interactions:

$$\vec{S}_1 \cdot \vec{S}_2 \qquad \vec{L} \cdot \vec{S} \qquad S_{12} \tag{3.13}$$

where S_{12} is the tensor interaction. Let us start by examining the spin-dependent interactions of QED in the hydrogen atom.

3.2.1 The Spin-Orbit Interaction

For the hydrogen atom, from the point of view of the electron the proton circles around. This orbital motion crates a magnetic field at the centre given by

$$B = \frac{ev}{cr^2} \tag{3.14}$$

or in terms of the electron orbital angular momentum $L = mvr$

$$\vec{B} = \frac{e}{mcr^3}\vec{L} \tag{3.15}$$

The spinning electron constitutes a tiny magnetic dipole with dipole moment

$$\vec{\mu} = -\frac{e}{mc}\vec{S} \tag{3.16}$$

The energy of a magnetic dipole in the presence of a magnetic field $\vec{B}$ is

$$W = -\vec{\mu} \cdot \vec{B} \tag{3.17}$$

If we derive this more rigorously as a succession of infitesimal Lorentz transformations we obtain the Thomas precession, which introduces a factor of 1/2:

$$\Delta H_{SO} = \frac{e^2}{2m^2c^2r^3}\vec{L} \cdot \vec{S} \tag{3.18}$$

The expectation value of $\vec{L} \cdot \vec{S}$ is given by

$$\vec{L} \cdot \vec{S} = \frac{1}{2}[J^2 - L^2 - S^2] = \frac{1}{2}[j(j+1) - l(l+1) - s(s+1)] \tag{3.19}$$

which gives, for example,

$$\begin{array}{ll} {}^3P_2 & \vec{L} \cdot \vec{S} = 1 \\ {}^3P_1 & \vec{L} \cdot \vec{S} = -1 \\ {}^3P_0 & \vec{L} \cdot \vec{S} = -2 \end{array} \tag{3.20}$$

Figure 3.3: The spin orbit splitting for P-wave mesons

3.2.2 The Hyperfine Interaction

Again in hydrogen, the proton has a dipole moment

$$\vec{\mu}_p = \gamma_p \frac{e}{m_p c} \vec{S}_p \tag{3.21}$$

where γ_p is the proton diple moment in units of nucleon magnetons ($\gamma_p = 2.7928$).

The proton spin acts directly with the electron spin. The magnetic dipole, μ_p, has a field

$$\vec{B}(\vec{r}) = \underbrace{\frac{1}{r^3}\left[\frac{3(\vec{\mu}\cdot\vec{r})\vec{r}}{r^2} - \vec{\mu}\right]}_{r>a} + \underbrace{\frac{8\pi}{3}\vec{\mu}\delta^3(\vec{r})}_{r<a} \tag{3.22}$$

We can represent the two pieces pictorially by The energy of the electron in the presence of the dipole is

$$\Delta H_{SS} = \frac{\gamma_p e^2}{m m_p c^2}\left\{\frac{1}{r^3}[3(\vec{S}_p\cdot\hat{r})(\vec{S}_e\cdot\hat{r}) - \vec{S}_p\cdot\vec{S}_e] + \frac{8\pi}{3}(\vec{S}_p\cdot\vec{S}_e)\delta^3(\vec{r})\right\} \tag{3.23}$$

This gives rise to the hyperfine structure of hydrogen, in particular the 21cm line in hydrogen. The expectation value of the $\vec{S}_1\cdot\vec{S}_2$ is given by

$$\vec{S}_1\cdot\vec{S}_2 = \frac{1}{2}[S^2 - S_1^2 - S_2^2] = \frac{1}{2}[s(s+1) - \frac{3}{2} \tag{3.24}$$

Figure 3.4: The hyperfine interaction

Figure 3.5: The hyperfine splitting for S-wave states

3.2.3 The Spin-Dependent Potentials in Quarkonia

One can take the above results over to the one-gluon interactions in QCD.

$$H_{ij}^{hyp} = \frac{-\alpha_s(r)}{m_i m_j}\left\{\frac{8\pi}{3}\vec{S}_i\cdot\vec{S}_j\delta^3(r_{ij}) + \frac{1}{r_{ij}^3}\left[\frac{3\vec{S}_i\cdot\vec{r}_{ij}\vec{S}_j\cdot\vec{r}_{ij}}{r_{ij}^3} - \vec{S}_i\cdot\vec{S}_j\right]\right\} \tag{3.25}$$

$$H_{ij}^{S.O.(c.m.)} = \frac{-\alpha_s(r)}{r_{ij}^3}\left(\frac{1}{m_i}+\frac{1}{m_j}\right)\left(\frac{\vec{S}_i}{m_i}+\frac{\vec{S}_j}{m_j}\right)\cdot\vec{L}\ \ \vec{F}_i\cdot\vec{F}_j \tag{3.26}$$

$$H_{ij}^{S.O.(t.p.)} = -\frac{1}{2r_{ij}}\frac{\partial V(r)}{\partial r_{ij}}\left(\frac{\vec{S}_i}{m_i^2}+\frac{\vec{S}_j}{m_j^2}\right)\cdot\vec{L} \tag{3.27}$$

For mesons $\langle\vec{F}_i\cdot\vec{F}_j\rangle = -4/3$.

3.2.4 A simplified look at spin-dependent splittings in charmonium

Let us examine the spin-dependent splittings in the $c\bar{c}$ system. Using harmonic oscillator wavefunctions simplifies the calculations. We use harmonic oscillator wavefunctions with the oscillator parameters fitted to the r.m.s. radii of wavefunctions which were found by solving the Schrodinger equation for a linear plus Coulomb potential. This approximation gives reasonably good results.

$$\psi_{1S} = \frac{2}{\pi^{1/4}}\beta^{3/2}e^{-\beta^2r^2/2}Y_{00} \tag{3.28}$$

where $\langle r^2\rangle_{1S} = \frac{3}{2}\frac{1}{\beta^2} = 2.5\ \mathrm{GeV}^{-2}$ so $\beta_{1S} = 0.77$ GeV.

$$\psi_{2S} = \sqrt{\frac{8}{3}}\frac{\beta^{3/2}}{\pi^{1/4}}(\frac{3}{2}-\beta^2r^2)e^{-\beta^2r^2/2}Y_{00} \tag{3.29}$$

where $\langle r^2\rangle_{2S} = \frac{7}{2}\frac{1}{\beta^2} = 11.0\ \mathrm{GeV}^{-2}$ so $\beta_{2S} = 0.564$ GeV.

$$\psi_{1P} = \sqrt{\frac{8}{3}}\frac{\beta^{5/2}r}{\pi^{1/4}}e^{-\beta^2r^2/2}Y_{1m} \tag{3.30}$$

where $\langle r^2\rangle_{1S} = \frac{5}{2}\frac{1}{\beta^2} \simeq 7.0$ GeV^{-2} so $\beta_{1P} = 0.598$ GeV, $\langle 1/r\rangle_{1P} = \frac{4}{3}\frac{\beta}{\pi^{1/2}} = 0.45$ GeV, and $\langle 1/r^3\rangle_{1P} = \frac{4}{3}\frac{\beta^3}{\pi^{1/2}} = 0.16$ GeV. With these wavefunctions we can calculate the spin-dependent splittings in the S-waves and P-wave charmonium mesons.

Hyperfine Effects

For the S-waves the Hyperfine effects are given by

$$H_{ij}^{hyp} = \frac{32\pi}{9}\frac{\alpha_s(r)}{m_c^2}\vec{S}_1 \cdot \vec{S}_2 \delta^3(r_{ij}) \tag{3.31}$$

From above we have $\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}[s(s+1) - \frac{3}{2}]$ which gives

$$\Rightarrow \begin{array}{ll} \langle {}^3S_1|\vec{S}_1 \cdot \vec{S}_2|{}^3S_1\rangle & = +1/4 \\ \langle {}^1S_0|\vec{S}_1 \cdot \vec{S}_2|{}^1S_0\rangle & = -3/4 \end{array} \tag{3.32}$$

Substituting into H_{ij} gives

$$\begin{aligned} M({}^3S_1) - M({}^1S_0) &= \frac{32\pi}{9}\frac{\alpha_s}{m^2}\langle\delta^3(r_{ij})\rangle \\ &= \frac{32\pi}{9}\frac{\alpha_s}{m^2}|\psi(0)|^2 \\ &= \frac{32\pi}{9}\frac{\alpha_s}{m^2}\frac{\beta^3}{\pi^{3/2}} \\ &= 0.115 \text{ GeV} \end{aligned} \tag{3.33}$$

where we took $\beta = 0.77$ GeV, $\alpha_s = 0.32$, and $m_c = 1.6$ GeV. The experimental value is 115 MeV. Repeating the excercise for the $2S$ states we obtain $M(2^3S_1) - M(2^1S_0) = 67$ MeV. We can obtain a crude estimate of the $\Upsilon - \eta_b$ splitting if we assume the wavefunction at the origin is the same in both cases:

$$M(\Upsilon) - M(\eta_b) \simeq \left(\frac{m_c^2}{m_b^2}\right) \times (M(J/\psi) - M(\eta_c)) \tag{3.34}$$

Using $m_b \simeq 5$ GeV we obtain$M(\Upsilon) - M(\eta_b) \simeq 11$ MeV.

Fine Structure

We can write the ^3P_J masses as

$$\begin{array}{lllll} M = & M(1P) & +a\langle\vec{L}\cdot\vec{S}\rangle & +b\langle S_{12}\rangle & \\ M(^3P_2) = & M(1P) & +a & -\frac{2}{5}b & = 3556 \\ M(^3P_1) = & M(1P) & -a & +2b & = 3511 \\ M(^3P_2) = & M(1P) & -2a & -4b & = 3415 \end{array} \tag{3.35}$$

where a and b are given by evaluating the H^{SO} for the harmonic oscillator wavefunctions. For example, given

$$H_{ij}^{S.O.(CM)} = \frac{4}{3}\frac{\alpha_s}{r^3}\frac{2}{m_c^2}\vec{S}\cdot\vec{L} \tag{3.36}$$

and

$$H_{ij}^{S.O.(TP)} = -\frac{1}{2r}\frac{\partial V}{\partial r}\frac{1}{m_c^2}\vec{S}\cdot\vec{L} \tag{3.37}$$

with

$$V = -\frac{4}{3}\frac{\alpha_s}{r} + br \tag{3.38}$$

we need the expectation values

$$\langle\frac{1}{r^3}\rangle = \frac{4}{3}\frac{\beta^3}{\pi^{1/2}} \tag{3.39}$$

and

$$\langle\frac{1}{r}\rangle = \frac{4}{3}\frac{\beta}{\pi^{1/2}} \tag{3.40}$$

We therefore obtain from the one gluon exchange piece

$$a = \frac{3}{2m_c^2}\frac{4}{3}\frac{\alpha_s}{r^3} = 40 \text{ MeV} \tag{3.41}$$

and

$$b = \frac{1}{4m_c^2}\frac{4}{3}\frac{\alpha_s}{r^3} = 7 \text{ MeV} \tag{3.42}$$

and from the confining piece a contribution to a of -16 MeV. This results in the masses $M(^3P_2) = 3556$ MeV, $M(^3P_1) = 3505$ MeV, and $M(^3P_0) = 3424$ MeV in reasonable agreement with the experimental numbers.

Figure 3.6: Electromagnetic transitions in mesons.

3.3 Radiative Transitions

It's one thing to be able to predict the masses of the various hadronic states but we would also like to probe the internal structure of these states. We can do so by studying their decays which will be sensitive to the internal structure. By far the theoretically cleanest decays are electromagnetic transitions. The electromagnetic interaction is well known and we are confident that perturbation theory works. The physics of radiative transitions is exactly the same as the physics of radiative transitions in atomic or nuclear physics.

An electromagetic transition is described by the single quark transitions;

3.3.1 Phase Space

Before we study the transition matrix element lets first look at the two body phase space relevant to a radiative transition, $M_i \to M_f \gamma$:

$$d\Gamma = \frac{(2\pi)^4 \delta^4(p_f + p_\gamma - p_i)}{2M_i} |M_{fi}|^2 \frac{d^3 p_f}{(2\pi)^3 (2E_f)} \frac{d^3 p_\gamma}{(2\pi)^3 (2E_\gamma)} \tag{3.43}$$

integrating we obtain:

$$\Gamma = \frac{1}{8M_i} \frac{1}{(2\pi)^2} \int |M_{if}|^2 \delta(E_f + E_\gamma - E_i) \delta^3(\vec{P}_f - \vec{p}_\gamma) \frac{d^3 p_f}{\sqrt{\vec{p}_f^2 + m_f^2}} \frac{d^3 p_\gamma}{\sqrt{\vec{p}_\gamma}} \tag{3.44}$$

$$= \frac{1}{8M_i}\frac{1}{(2\pi)^2}\int |M_{if}|^2\delta(E_f+E_\gamma-M_i)\frac{d^3p_f}{p_f\sqrt{p_f^2+m_f^2}} \tag{3.45}$$

$$= \frac{1}{8\pi M_i}\int |M_{if}|^2\delta(M_i-\sqrt{p^2+m_f^2}-p)\frac{p^2dp}{p\sqrt{p^2+m_f^2}} \tag{3.46}$$

where I used the notation that $\vec{p}_f^2 = p_f^2 = p^2$. Next I use the substitutions

$$\begin{aligned} E &= \sqrt{\vec{p}_f^2+m_f^2}+\sqrt{\vec{p}_f^2} \\ dE &= \frac{pdp}{\sqrt{p_f^2+m_f^2}}+\frac{pdp}{p} \\ &= \left(\frac{p+\sqrt{p^2+m^2}}{p\sqrt{p^2+m^2}}\right)pdp \end{aligned} \tag{3.47}$$

so that

$$dE = \frac{E}{\sqrt{p^2+m^2}}dp \Rightarrow \frac{dp}{\sqrt{p^2+m^2}} = \frac{dE}{E} \tag{3.48}$$

so that

$$\begin{aligned} \Gamma &= \frac{1}{8\pi M_i}\int_{m^2}^{\infty}|M_{if}|^2\delta(M_i-E)\frac{p}{E}dE \\ &= \frac{1}{8\pi M_i^2}|M_{if}|^2p \end{aligned} \tag{3.49}$$

where

$$p = \frac{1}{2M}\sqrt{M_i^4+M_f^4-2M_i^2M_f^2} = \frac{M_i^2-M_f^2}{2M_i} \tag{3.50}$$

This can rearranged further to obtain

$$\Gamma = \frac{|M_{if}|^2}{8\pi M_i}(1-M_f^2/M_i^2) \tag{3.51}$$

and

$$\frac{d\Gamma}{d\cos\theta} = \frac{|M_{if}|^2}{16\pi^2M_i}(1-M_f^2/M_i^2) = \frac{|M_{if}|^2}{8\pi^2M_i^2}\omega \tag{3.52}$$

where $\omega = k_\gamma(=p_\gamma)$.

3.3.2 E1 Transitions

Lets start with E1 transitions for which we can resort to nonrelativistic quantum mechanics. Using minimal substitution

$$\frac{p^2}{2m} \to \frac{(\vec{p} - e\vec{A})^2}{2m} = \frac{p^2}{2m} - \frac{e\vec{p}\cdot\vec{A}}{2m} - \frac{e\vec{A}\cdot\vec{p}}{2m} + e^2\frac{\vec{A}^2}{2m} \tag{3.53}$$

where A is the electromagnetic field and $p^2/2m$ is the kinetic energy term in the Schrodinger equation. Since the $e^2\vec{A}^2$ is higher order in e and therefore smaller we drop it. We are interested in the interaction term

$$H_I = -\frac{e}{2m}(\vec{A}\cdot\vec{p} + \vec{p}\cdot\vec{A}) \tag{3.54}$$

The photon wavefunction is given by

$$\vec{A}(x) = \frac{1}{\sqrt{2\omega}}\vec{\epsilon}(\vec{k})e^{i\vec{k}\cdot\vec{x}} \tag{3.55}$$

where $\vec{\epsilon}$ is the photon polarization vector. We can expand the exponential

$$e^{i\vec{k}\cdot\vec{x}} \simeq 1 + i\vec{k}\cdot\vec{x} + \ldots \tag{3.56}$$

in the long wavelength limit we have $1/k >> r$ where r is the size of the hadron. Therefore we can approximate the photon wavefunction with

$$\vec{A}(x) \simeq \frac{1}{\sqrt{2\omega}}\vec{\epsilon}(\vec{k}) \tag{3.57}$$

Substituting into H_I gives

$$H_I = -\frac{e}{2m}(\vec{\epsilon}\cdot\vec{p} + \vec{p}\cdot\vec{\epsilon}) \tag{3.58}$$

The next step is to evaluate the expectation value of this operator. We start by using the commutation relations

$$[p_i, r_j] = -i\delta_{ij} \tag{3.59}$$

which gives

$$[\vec{p}^2, r_j] = p_i[p_i, r_j] + [p_i, r_j]p_i = -2ip_j \tag{3.60}$$

or

$$p_j = \frac{i}{2}[p^2, r_j] \tag{3.61}$$

Substituting

$$\begin{aligned}
\langle A|p_i|B\rangle &= i\langle A|[\vec{p}^2/2, r_j]|b\rangle \\
&= i\mu\langle A|[H, r_j]|B\rangle \\
&= i\mu\langle A|Hr_j - r_jH|B\rangle \\
&= i\mu(E_A - E_B)\langle A|r_j|B\rangle \\
&= i\frac{m}{2}\omega\langle A|r_j|B\rangle
\end{aligned} \tag{3.62}$$

Where the reduced mass, $\mu = m/2$ and $\omega = E_A - E_B$. We used the fact that for $H = p^2/2\mu + V(r)$ $[V(r), r] = 0$. We can subsitute this result into the expectation value for H_I to obtain

$$\begin{aligned}
\langle A|H_I|B\rangle &= -\frac{iem\omega}{2m}\langle A|r_i|B\rangle\epsilon_i \\
&= -\frac{ie\omega}{2}\langle A|r_i|B\rangle\epsilon_i \\
&= -\frac{ie\omega}{2}\langle A|\vec{r}|B\rangle \cdot \vec{\epsilon}
\end{aligned} \tag{3.63}$$

Given the matrix element and the phase space factors there are two approaches to evaluating the matrix elements;

Method 1

We start by summing over the photon polarizations and obtain the sum:

$$\sum_{pol} \vec{\epsilon}_i(k)\vec{\epsilon}^*(k) = \delta_{ij} - k_ik_j/\vec{k}^2 \tag{3.64}$$

so that

$$\sum_{pol} |\langle B|H_I|A\rangle|^2 = \omega^2 e^2 Q^2 \left\{ |\langle B|\vec{r}|A\rangle|^2 - |\langle B|\vec{r}\cdot\hat{k}|A\rangle|^2 \right\} \tag{3.65}$$

Averaging over directions gives the final result

$$= \omega^2 e^2 Q^2 \frac{2}{3} |\langle B|\vec{r}|A\rangle|^2 \tag{3.66}$$

Let us start with the transition ${}^3P_J \to^3 S_1\gamma$ to start with. The orbital angular momentum is zero in the final state. We can choose any value of J_Z since we averaged over the photon directions. It is convenient to choose $J_Z = J$. Before proceeding we need to write down the form for the meson wavefunction;

$$|M\rangle = \sqrt{2M}\psi(r) \tag{3.67}$$

where the factor of $\sqrt{2M}$ is introduced so that the wavefunctions are normalized to one when integrating over the relativisitic phase space integral. For the ${}^3P_2$ state with $J_Z = 2$ we have

$$\begin{aligned} |J = J_Z = 2\rangle &= |L = L_Z = 1\rangle \otimes |S = S_Z = 1\rangle \\ &= |Y_{11} \uparrow\uparrow\rangle \end{aligned} \tag{3.68}$$

Only the $J'_Z = S'_Z = 1$ contributes since H_I does not flip the spin. Putting this together and evaluating the matrix element we obtain

$$\begin{aligned} \langle f|\vec{r}|i\rangle &= \langle f|r|i\rangle \int \langle Y_{00} \uparrow\uparrow | \sqrt{\frac{4\pi}{3}} Y_{1-1} | Y_{11} \uparrow\uparrow\rangle d\Omega \\ &= \langle f|r|i\rangle \sqrt{\frac{1}{3}} \end{aligned} \tag{3.69}$$

$$\tag{3.70}$$

where

$$\langle f|r|i\rangle = \int r^2 dr R_f(r) r R_i(r) \sqrt{2M_i}\sqrt{2M_f} \tag{3.71}$$

Therefore

$$\begin{aligned} \Gamma({}^3P_2 \to^3 S_1) &= \frac{1}{8\pi M^2}|M_{if}|^2\omega \\ &= \frac{\omega}{8\pi M^2}\omega^2 e^2 Q^2 |\langle f|r|i\rangle|^2 (sM_i)(2M_f) \times \frac{2}{3} \times \frac{1}{3} \\ &= \frac{4\pi\alpha\omega^3 e_q^2}{8\pi}\frac{8}{9}|\langle f|r|i\rangle|^2 \left(\frac{M_i M_f}{M_i M_i}\right) \\ &= \frac{4}{9}\alpha\omega^3 e_q^2 |\langle f|r|i\rangle|^2 \left(\frac{M_f}{M_i}\right) \end{aligned} \tag{3.72}$$

We repeat the excercise for ${}^3P_1 \rightarrow^3 S_1$

$$|J = J_Z = 1\rangle = \frac{1}{\sqrt{2}}|Y_{11}\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) - Y_{10} \uparrow\uparrow\rangle \qquad (3.73)$$

so that

$$\begin{aligned}\langle L' = 0|\vec{r}|J = J_Z = 1\rangle &= \frac{1}{\sqrt{2}}\langle L' = 0|\vec{r}|L = L_Z = 1\rangle - \frac{1}{\sqrt{2}}\langle L' = 0|\vec{r}|L = L_Z = 0\rangle \\ &= \left[\frac{1}{\sqrt{2}}\langle L' = 0|\frac{1}{\sqrt{3}}(-\frac{\hat{x} + i\hat{y}}{\sqrt{2}})|L = L_Z = 1\rangle - \frac{1}{\sqrt{2}}\langle L' = 0|\frac{1}{\sqrt{3}}\hat{z}|L = L_Z = 0\rangle\right]\langle\end{aligned}$$

so that

$$|\langle^3S_1|\vec{r}|^3P_1\rangle|^2 = [\frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{3}]|\langle 1S|r|1P\rangle|^2 \qquad (3.75)$$

Similarly for ${}^3P_0 \rightarrow^3 S_1$

$$|J = J_Z = 0\rangle = \sqrt{\frac{1}{3}}|Y_{11} \downarrow\downarrow - Y_{10}\sqrt{\frac{1}{2}}(\uparrow\downarrow + \downarrow\uparrow) + Y_{1-1} \uparrow\uparrow\rangle \qquad (3.76)$$

resulting in

$$|\langle^3S_1|\vec{r}|^3P_0\rangle|^2 = [\frac{1}{3}\frac{1}{3} + \frac{1}{3}\frac{1}{3} + \frac{1}{3}\frac{1}{3}]|\langle 1S|r|1P\rangle|^2 \qquad (3.77)$$

Summarizing all these results we obtain

$$\begin{aligned}\Gamma({}^3P_2 \rightarrow^3 S_1\gamma) &= \frac{\omega^3e^2Q^2}{3\pi}\frac{1}{3}|\langle 1S|r|1P\rangle|^2 \qquad (3.78)\\ \Gamma({}^3P_1 \rightarrow^3 S_1\gamma) &= \frac{\omega^3e^2Q^2}{3\pi}\left\{\frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{3}\right\}\frac{1}{3}|\langle 1S|r|1P\rangle|^2 \qquad (3.79)\\ \Gamma({}^3P_0 \rightarrow^3 S_1\gamma) &= \frac{\omega^3e^2Q^2}{3\pi}\left\{\frac{1}{3}\frac{1}{3} + \frac{1}{3}\frac{1}{3} + \frac{1}{3}\frac{1}{3}\right\}|\langle 1S|r|1P\rangle|^2 \qquad (3.80)\end{aligned}$$

Comparing these expressions we see that in all cases

$$\Gamma({}^3P_J \rightarrow^3 S_1\gamma) = \frac{4\alpha\omega^3Q^2}{9}|\langle 1S|r|1P\rangle|^2 \qquad (3.81)$$

Similarly we obtain:

$$\Gamma({}^3S_1 \rightarrow^3 P_J\gamma) = \frac{4\alpha\omega^3Q^2(2J+1)}{27}|\langle 1S|r|1P\rangle|^2 \qquad (3.82)$$

Let us return to our "effective" wavefunctions for the charmonium states:

$$\psi_{1S} = \frac{2}{\pi^{1/4}} \beta^{3/2} e^{-\beta^2 r^2/2} Y_{00} \qquad \beta = 0.77 \text{ GeV} \tag{3.83}$$

$$\psi_{1P} = \sqrt{\frac{8}{3}} \frac{\beta^{5/2} r}{\pi^{1/4}} e^{-\beta^2 r^2/2} Y_{1m} \qquad \beta = 0.598 \text{ GeV} \tag{3.84}$$

This gives

$$\begin{aligned} \langle \psi_{1S} | r | \psi_{1P} \rangle &= \frac{2}{\pi^{1/4}} \sqrt{\frac{8}{3}} \frac{1}{\pi^{1/4}} \beta_S^{3/2} \beta_P^{5/2} \int r^4 e^{-(\beta_S^2 + \beta_P^2) r^2/2} dr \\ &= \sqrt{\frac{8}{3}} 15 \frac{\beta_S^{3/2} \beta_P^{5/2}}{(\beta_S^2 + \beta_P^2)^{5/2}} \\ &= 5.2 \text{ GeV}^{-1} \end{aligned} \tag{3.85}$$

Plugging this in we obtain

$$\Gamma(^3P_2 \rightarrow ^3 S_1 \gamma) = 0.59 \text{ MeV} \quad vs \quad \Gamma^{expt} = 0.351^{+.2}_{-.14} \text{ MeV} \tag{3.86}$$

$$\Gamma(^3P_1 \rightarrow ^3 S_1 \gamma) = \text{ MeV} \quad vs \quad \Gamma^{expt} < 0.355 \text{ MeV} \tag{3.87}$$

Method 2

Another technique which is useful uses helicity amplitudes.

3.3.3 M1 Transitions and Magnetic Moments

Because quarks have spin they can omit a photon via a spin flip, the magnetic moment transition.

To obtain the interaction Hamiltonian we perform a non-relativistic reduction of the

$$H_I = e \int dx j^\mu_{em}(x) A_\mu(x) \tag{3.88}$$

where $j^\mu_{em}(x) = \bar{q}(x) Q \gamma^\mu q(x)$ is the electromagnetic current. We expand the Dirac spinors to lowest order in p/m. Denoting the large and small components by q_1 and q_2

$$q_2(x) = -\frac{i\vec{\sigma} \cdot \vec{\nabla}}{2m} q_1(x) \tag{3.89}$$

so that

$$\vec{j}_{em}(x) = \frac{-i}{2m}[q_1^\dagger Q(\nabla q_1) - (\nabla q_1^\dagger)Qq_1 + i\nabla \times q_1^\dagger Q\vec{\sigma}q_1] \tag{3.90}$$

and the interaction Hamiltonian is given by:

$$H_I = \frac{-eQ}{2m}[\vec{A}(\vec{r}) \cdot \vec{p} + \vec{p} \cdot \vec{A}(\vec{r}) + \vec{\sigma} \cdot [\vec{\nabla} \times \vec{A}(\vec{r})] \tag{3.91}$$

so

$$\langle 0|H_I|\gamma(\vec{k}, \epsilon)\rangle = -\frac{1}{(2\pi)^{3/2}}\frac{1}{(2\omega)^{1/2}}eQ\frac{1}{2m}[e^{i\vec{k}\cdot\vec{r}}\vec{\epsilon}\cdot\vec{p} + \vec{\epsilon}\cdot\vec{p}e^{i\vec{k}\cdot\vec{r}} + i\vec{\sigma}\cdot(\vec{k}\times\vec{\epsilon})e^{i\vec{k}\cdot\vec{r}}] \tag{3.92}$$

For antiquarks change the sign of the charge. We have already examined the first two terms in our discussion of the E1 transitions. The last term gives rise to the spin-flip M1 transitions.

$$\mu = \frac{e}{2m_q} \tag{3.93}$$

is the magnetic dipole moment of the quark. Therefore for the magnetic dipole transistions we have

$$M_{if} = i\mu\langle f|\vec{\sigma}|i\rangle \cdot \vec{k} \times \vec{\epsilon}^* \tag{3.94}$$

where the photon polarization vector is given by

$$\vec{\epsilon} = \frac{1}{\sqrt{2}}(1, \pm i, 0) \tag{3.95}$$

Thus,

$$\begin{vmatrix} \sigma_x & \sigma_y & \sigma_z \\ k_x & k_y & k_z \\ 1 & i & 0 \end{vmatrix} = i\sigma_z(k_x + ik_y) - ik_z\sigma_x + k_z\sigma_y \tag{3.96}$$

So $\vec{\sigma} \cdot (\vec{k} \times \vec{\epsilon}_\pm) = \mp\frac{ik_z}{\sqrt{2}}(\sigma_x \pm i\sigma_y) \pm \frac{i\sigma_z}{\sqrt{2}}(k_x \pm ik_y)$. Choosing z as the γ directions, and assuming one helicity state, we obtain

$$M_{if} = -\frac{ie_q}{2m}k_\gamma\langle f|\sigma_x - i\sigma_y|i\rangle \tag{3.97}$$

where $\sigma_x - i\sigma_y = \sigma_-$. If instead we took y as the photon direction we would obtain

$$\begin{aligned} M_{if} &= \frac{ie_q}{2m} k_\gamma \langle f|\sigma_z|i\rangle && (3.98) \\ &= k_\gamma \sqrt{2M_i}\sqrt{2M_f} \int d^3r \psi_f^*(r)\psi_i(r) \times \langle f| \sum \mu_i \sigma_{zi} |i\rangle && (3.99) \end{aligned}$$

Once again we can return to our simplified charmonium system and look at the M1 transition $J/\psi \to \eta_c\gamma$ which is a ${}^3S_1 \to^1 S_0\gamma$ transition. The amplitude for this decay is given by

$$\begin{aligned} A({}^3S_1 \to^1 S_0\gamma) &= -ik_\gamma \sqrt{2M_i}\sqrt{2M_f}\langle f|i\rangle \\ &\quad \times \langle \sqrt{\frac{1}{2}}(\uparrow\downarrow - \downarrow\uparrow)| \frac{e_q}{2m_q}\frac{(\sigma_x - i\sigma_y)_q}{\sqrt{2}} + \mu_{\bar{q}} \frac{(\sigma_x - i\sigma_y)_{\bar{q}}}{\sqrt{2}} && (3.100) \\ &= -ik_\gamma \sqrt{2M_i}\sqrt{2M_f}\langle f|i\rangle \left[\frac{-e_q}{2m_q} + \frac{e_{\bar{q}}}{2m_{\bar{q}}}\right] && (3.101) \\ &= -ik_\gamma \sqrt{2M_i}\sqrt{2M_f}\langle f|i\rangle \frac{ee_q}{m_c} && (3.102) \end{aligned}$$

Squaring and including the phase space factors gives the differential width:

$$\frac{d\Gamma}{d\Omega} = k_\gamma \frac{4\pi\alpha}{8\pi^2} k_\gamma^2 |\langle f|i\rangle|^2 \frac{e_c^2}{m_c^2} \tag{3.103}$$

and averaging over angles gives the total width

$$\Gamma = \frac{k_\gamma^3}{3\pi} |\langle f|i\rangle|^2 \frac{e_c^2}{m_c^2} \tag{3.104}$$

We take $\langle f|i\rangle = 1$ which is reasonable in first order perturbation theory. $k_\gamma = 115$ MeV which gives us the result $\Gamma = 0.19$ MeV vs the experimental number of 0.88 keV. This is not too good agreement. What about the transition $\psi' \to \eta_c\gamma$ which is $2^3S_1 \to 1^1S_0$. In this case we would expect the spatial overlap integral to be equal to zero since the $1S$ and $2S$ wavefunctions are orthogonal. Nevertheless the transition is observed. There are a number of reasons for this. In second order perturbation theory the hyperfine interaction which split the singlet and triplet states would also change the wavefunctions so that the 1^3S_1

Figure 3.7: Leptonic decays of vector mesons.

and 1^1S_0 are no longer identical and the 2^3S_1 wavefunction is not necessarily orthogonal to the 1^1S_0 wavefunction. In addition we used the long wavelength approximation. This is not a bad approximation but when the states we are considering are orthogonal the next term in the expansion will give a small but nonzero contribution to the overlap integral.

3.4 Leptonic Decays

We can also examine other decays in the quark model such as $J/\psi \to e^+e^-$ ($^3S_1 \to e^+e^-$) which is shown if Fig. ??. The hadronic part of the matrix element for this decay is given by

$$\langle 0|j^\mu_{em}|V(\uparrow\uparrow)\rangle = \sqrt{3\times 2M}\int d^3p\phi_s(p)Y_{00}\langle 0|j^\mu_{em}|c\bar{c}\rangle \tag{3.105}$$

where the factor of $\sqrt{3}$ comes form colour:

$$\sum_{colour} = \sqrt{\frac{1}{3}}(r\bar{r}+b\bar{b}+g\bar{g}) = \frac{3}{\sqrt{3}} = \sqrt{3} \tag{3.106}$$

If we go through the entire calculation we obtain in the end the expression

$$\Gamma = \frac{16\pi\alpha^2 e_q^2}{M^2}|\psi(0)|^2 \tag{3.107}$$

Figure 3.8: Annihilation decays of quarkonium to photons and gluons.

The wavefunction at the origin reflects that the quark and antiquark must annihilate which is proportional to the probability that they will be at zero separation.

3.5 Other Decays

We can study many other decays of quarkonium which are of the form of annihilations to photons and gluons. The results can be taken over from positronium annihilation except when gluons are involved we must be careful to include the correct colour factors.

For the case of gluons, at least two must be produced as a single gluon has colour which would violate colour confinement. Two gluons can combine to give a colour singlet. Only $C = +$ states can decay to two gluons or photons. $C = -$ states must decay to an odd number of gluons and photons to conserve C-parity. For example, a ${}^3S_1$ state can decay to three gluons, two gluons plus a photon, or three photons.

Problems

1. I have solved the Schrodinger equation for the $b\bar{b}$ system with the linear plus Coulomb potential and have approximated the exact wavefunctions with these approximations:

$$\psi_{1S} = \frac{2}{\pi^{1/4}}\beta^{3/2}e^{-\beta^2r^2/2}\,Y_{00}$$

$$\psi_{2S} = \sqrt{\frac{8}{3}}\frac{\beta^{3/2}}{\pi^{1/4}}\left(\frac{3}{2} - \beta^2r^2\right)e^{-\beta^2r^2/2}\,Y_{00}$$

$$\psi_{3S} = \sqrt{\frac{32}{15}}\frac{\beta^{3/2}}{\pi^{1/4}}\left(\frac{15}{8} - \frac{5}{2}\beta^2r^2 + \frac{1}{2}\beta^4r^4\right)e^{-\beta^2r^2/2}\,Y_{00}$$

$$\psi_{1P} = \sqrt{\frac{8}{3}}\frac{\beta^{5/2}r}{\pi^{1/4}}e^{-\beta^2r^2/2}\,Y_{1m}$$

$$\psi_{2P} = \sqrt{\frac{16}{15}}\frac{\beta^{5/2}r}{\pi^{1/4}}\left(\frac{5}{2} - \beta^2r^2\right)e^{-\beta^2r^2/2}\,Y_{1m}$$

$\beta_{1S} = 1.2$ GeV, $\beta_{2S} = 0.83$ GeV, $\beta_{3S} = 0.72$ GeV, $\beta_{1P} = 0.89$ GeV, and $\beta_{2P} = 0.73$ GeV. In what follows take $m_b = 5.0$ GeV, $\alpha_s = 0.4$, and $b = 0.18$ GeV^2.

The following integrals might be useful:

$$\int_0^\infty x^{2n}e^{-px^2}dx = \frac{(2n-1)!!}{2(2p)^n}\sqrt{\frac{\pi}{p}} \quad \text{and} \quad \int_0^\infty x^{2n+1}e^{-px^2}dx = \frac{n!}{2p^{n+1}}$$

where $(2n+1)!! = 1.3...(2n+1)$.

(a) Calculate the hyperfine splitting for the 1S, 2S, and 3S $b\bar{b}$ states (using 1st order perturbation theory).

(b) Calculate the expectation values of $\langle \vec{L} \cdot \vec{S} \rangle$ for the P-wave mesons. Using these results and $\langle {}^3P_2|S_{12}|{}^3P_2 \rangle = -\frac{1}{10}$, $\langle {}^3P_1|S_{12}|{}^3P_1 \rangle = \frac{1}{2}$, and $\langle {}^3P_0|S_{12}|{}^3P_0 \rangle = -1$ calculate the spin dependent splittings in the 1P and 2P multiplets. Compare your results with the experimental splittings.

2. Let us continue with the previous problem. Refer there for the $b\bar{b}$ wavefunctions.

(a) Calculate the partial width for the magnetic dipole transitions for $\Upsilon(1S) \to \eta_b(1S)$. How many events would this give in a 10^6 $\Upsilon(1S)$ sample?

(b) Calculate the $\Upsilon(2S) \to \chi_b(1P)$ and $\Upsilon(3S) \to \chi_b(2P)$ E1 transition partial widths. Use the photon energies from the particle data book or from the measured masses. Compare to experimental values.

(c) Calculate the ration of the $\Gamma(\Upsilon(1S) \to e^+e^-) : \Gamma(\Upsilon(2S) \to e^+e^-) : \Gamma(\Upsilon(3S) \to e^+e^-)$. Compare to the exprimental values.

(d) Overall, how well does this simple calculation work. Can you explain discrepancies?

Chapter 4

Light Meson Spectroscopy

So far we have looked at heavy quarkonium spectroscopy where I have argued that the quark model has some connection to QCD. On the other hand, historically, it was the successes of the quark model in the light quark hadrons that led many physicists to believe the quark model has something to do with reality. In what follows we will see that the quark model does indeed describe the light meson spectroscopy but there are many puzzles.

Let us begin by considering the figure

What we see in this figure is that there is a smooth evolution going from the heavy $b\bar{b}$ system to the relativistic light-quark systems. Qualitatively we see the smae structure in the heavy and light systems although in the light systems the relativistic (spin-dependent) splittings are comparible to the orbital splittings. It is therefore important that relativistic effects are included.

In the previous chapter we studied the heavy quarkonium spectrum in detail. In figure – we show the spectrum, including transitions, for the strange mesons. We see that the spectrum is qualitatively the same as the charmonium spectrum. The main difference is that for the strange mesons many more orbitally excited states have been seen. This is a consequence of the production mechanisms used to study the strange mesons, $Kp \to K^* + p$ scattering which can excite many different quantum numbers, versus the production mechanism for the charmonium states, $e^+e^- \to \psi$ via an intermeditate photon which couples only to $J^{PC} = 1^{--}$ states. Thus, the study of light mesons examines

Figure 4.1: The evolution of the meson splittings from the heaviest $b\bar{b}$ states to the lightest $u\bar{u}$ states

properties complementary to those studied in the heavy quarkonium systems.

We begin by writing down the flavour wavefunctions for the light quark mesons. The flavour wavefunctions for the fundamental doublets are given by:

$$\begin{array}{rcl} u = |\frac{1}{2}\frac{1}{2}\rangle & \qquad & d = |\frac{1}{2} - \frac{1}{2}\rangle \\ \bar{d} = -|\frac{1}{2}\frac{1}{2}\rangle & \qquad & \bar{u} = |\frac{1}{2} - \frac{1}{2}\rangle \end{array}$$

Note that there is a phase for the "up" component of the antiquark doublet. This is because the antiquark doublet is not equivalent to the quark doublet. The flavour wavefunctions are given by:

$$\begin{aligned} |\rho^+\rangle,\ |\pi^+\rangle &= -|u\bar{d}\rangle \\ |\rho^0\rangle,\ |\pi^0\rangle &= \frac{1}{\sqrt{2}}|u\bar{u} - d\bar{d}\rangle \\ |\eta\rangle &= \frac{1}{\sqrt{6}}|u\bar{u} + d\bar{d} - 2s\bar{s}\rangle \\ |\eta'\rangle &= \frac{1}{\sqrt{3}}|u\bar{u} + d\bar{d} + s\bar{s}\rangle \\ |\omega\rangle &= \frac{1}{\sqrt{2}}|u\bar{u} + d\bar{d}\rangle \end{aligned}$$

Figure 4.2: The level diagram for the strange mesons.

$$\begin{aligned} |\phi\rangle &= |s\bar{s}\rangle \\ |K^+\rangle &= |u\bar{s}\rangle \\ |K^0\rangle &= d\bar{s}\rangle \\ |\bar{K}^0 &= -|s\bar{d}\rangle \\ |K^-\rangle &= |s\bar{u}\rangle \end{aligned}$$

In heavy quarkonium we used the Hamiltonian

$$H = \frac{p^2}{2\mu} + V(r) \tag{4.1}$$

with

$$V(r) = -\frac{4}{3}\frac{\alpha_s(r)}{r} + br \tag{4.2}$$

and the meson mass is given by $M = m_q + m_{\bar{q}} + E$. This is a non-relativistic formula. For a quark $(v/c) \simeq \langle p_E \rangle$ and we obtain the values:

System	v/c
$b\bar{b}$	0.26
$c\bar{c}$	0.45
$s\bar{s}$	0.78
$u\bar{u}$	0.9

So clearly the non-relativistic approximation is no longer valid. What should we do?

- Use it anyway and see what happens. Taking this approach the general features are O.K.

Figure 4.3: Annihilation mixing in selfconjugate mesons.

- Try to relativize it.

4.1 Annihilation Mixing

In the isoscalar mesons there is an additional complication which we have so far ignored but which plays an important role in the light quark mesons: Although one could calculate these in perturbation theory in the low Q^2 regime perturbation theory is not really valid. Therefore we just parametrize the mixing to reflect the data. We start by writing the mass matrix:

$$\begin{bmatrix} M_{u\bar{u}}+A & A & A \\ A & M_{d\bar{d}}+A & A \\ A & A & M_{s\bar{s}}+A \end{bmatrix} \begin{pmatrix} u\bar{u} \\ d\bar{d} \\ s\bar{s} \end{pmatrix} \tag{4.3}$$

The nature of the mixing depends on the relative strenght of the annihilation, A.

1. If $A = 0$ there is no mixing between the strange and non-strange states. This is equivilent to a 35.3^o mixing angle between the SU(3) singlet and octets and is sometimes referred to as ideal mixing.

2. If A is small and positive the isoscalar meson is slightly heavier than the isovector state. For example

$$M_\omega > M_\rho \tag{4.4}$$

3. If A is small and negative the isovector is slightly heavier than the isoscalar. For example

$$f_2 < a_2 \tag{4.5}$$

4. If A is large and positive there is complete mixing between the nonstrange and strange sectors. In this case the isoscalar is much heavier than the isovector. An example of this case are the pseudoscalar mesons where $M_\eta >> M_\pi$ and the flavour mixing is complete.

4.2 Spin Dependent Interactions

We take over the expressions we introduced for heavy quarkonium and ignore the relativistic corrections except where it is occasionally useful to comment on the differences they introduce.

4.2.1 $\vec{S}_1 \cdot \vec{S}_2$

The ${}^3S_1 -{}^1 S_0$ splitting is given by

$$\Delta(M({}^3S_1) - M({}^1S_0)) = \frac{3\pi\alpha_s}{9m_1m_2}|\psi(0)|^2 \tag{4.6}$$

This gives rise to the vector-pseudoscalar splittings:

$\rho - \pi$
$K^* - K$
$D^* - D$
$B^* - B$
$\psi - \eta_c$

We could approximate the ${}^3S_1$ and ${}^1S_0$ masses by

$$M({}^3S_1) = M(S) + \frac{1}{4}\frac{a}{m_q m_{\bar{q}}} \tag{4.7}$$

$$M({}^1S_0) = M(S) - \frac{3}{4}\frac{a}{m_q m_{\bar{q}}} \tag{4.8}$$

where a is a parameter which replaces the terms in the above equation. If a is approximately a constant then

$$\frac{M(\rho)-M(\pi)}{M(K^*)-M(K)} \simeq \frac{m_u m_s}{m_u m_u} \simeq \frac{m_s}{m_u} \simeq \frac{500}{300} \simeq 1.7 \tag{4.9}$$

$$\frac{770-140}{892-495} \simeq \frac{630}{400} \simeq 1.7 \tag{4.10}$$

Similarly

$$\frac{M(K^*)-M(K)}{M(D^*)-M(D)} \simeq \frac{m_u m_c}{m_u m_s} \simeq \frac{m_c}{m_s} \simeq \frac{1.6}{0.55} \simeq 2.9 \tag{4.11}$$

$$\frac{892-494}{2010-1870} \simeq \frac{400}{140} \simeq 2.9 \tag{4.12}$$

In general we find that the $^3S_1 -^1 S_0$ splittings are well described.

In addition the fact that the ^3P_J centre of gravity and the $^1P_1$ splitting is not large indicates that the $\vec{S}_1 \cdot \vec{S}_2$ interaction is short range with the properties expected from one-gluon-exchange.

4.2.2 $\vec{L} \cdot \vec{S}$ and the Long-Distance Potential

4.3 Electromagnetic Transitions

As before

$$\Gamma_{M1} = \frac{k_\gamma^3}{3\pi} |\langle f|i\rangle|^2 |\sum \mu_i \sigma_{zi}|^2 \tag{4.13}$$

We look at a couple of examples:

$K^{*+} \to K^+\gamma$

$$\begin{aligned} & \langle u\bar{s}\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)|\frac{e_i}{2m_i}\sigma_z|u\bar{s}(\uparrow\downarrow + \downarrow\uparrow)\rangle \\ = \; & \frac{1}{2}\langle u\bar{s}|\frac{e_q}{2m_q} + \frac{e_q}{2m_q} - \frac{e_{\bar{q}}}{2m_{\bar{q}}} - \frac{e_{\bar{q}}}{2m_{\bar{q}}}|u\bar{s}\rangle \\ = \; & \frac{1}{2}\left[\frac{e_u}{m_u} - \frac{e_s}{m_s}\right] \\ = \; & \frac{1}{2}\left[\frac{2}{3}\frac{1}{m_u} - \frac{1}{3}\frac{1}{m_s}\right] \end{aligned} \tag{4.14}$$

$\omega \to \pi^0 + \gamma$

$$\begin{aligned}
\langle \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)|\sigma_z^q|\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)\rangle &= 1 \\
\langle \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)|\sigma_z^{\bar{q}}|\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)\rangle &= -1
\end{aligned}$$

$$\begin{aligned}
\pi^0 &= \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \\
\omega^0 &= \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})
\end{aligned}$$

So

$$\begin{aligned}
\langle \pi^0 | \sum \mu_i \sigma_{zi} | \omega^0 \rangle &= \frac{1}{2}\mu_u - \frac{1}{2}\mu_d + \frac{1}{2}\mu_u - \frac{1}{2}\mu_d \\
&= \mu_u - \mu_d \qquad (4.15)
\end{aligned}$$

Putting these matrix elements together with the phase space factors we obtain:

$$\begin{aligned}
\Gamma(\omega^0 \to \pi^0\gamma) &= \frac{\omega^3}{3\pi}|\frac{1}{2}\frac{2}{3}\frac{e}{m_u} + \frac{1}{2}\frac{e}{3m_d}|^2|\langle f|i\rangle|^2 \\
&= \frac{\omega^3}{3\pi}\frac{4\pi\alpha}{4m_u^2} \\
&=
\end{aligned}$$

In addition, we could look at higher relativistic terms which give rise to higher multipoles:

	$a_2 \to \pi\gamma$	$M2$
to higher multipoles:	$a_1 \to \pi\gamma$	$E1$
	$b_1 \to \pi\gamma$	$E1$

4.4 Strong (Zweig Allowed) Decays

4.5 Missing Resonances

4.6 Puzzles in Light Meson Spectroscopy

Problems

1. The magnetic dipole decays of the u,d,s mesons are now fairly well known experimentally. The transition magnetic moments for such decays are proportional to

$$\langle f|\frac{e_i}{2m_i}\sigma_{iz}|i\rangle$$

between initial and final states with $s_z = 0$. $\frac{e_i}{2m_i}$ is the magnetic moments of quark i with spin $\frac{1}{2}\sigma_i$ and mass m_i; in your calculation use the standard values $m_u \approx m_d \approx 0.33$ GeV and $m_s \approx 0.55$ GeV. Compare your theoretical values with the experimental results:

4 Moment experiment (in nuclear magnetons) $\mu(\rho \to \pi)$ 0.67 ± 0.04 $\mu(\rho \to \eta)$ 1.7 ± 0.2 $\mu(\eta' \to \rho)$ 1.5 ± 0.3 $\mu(\omega \to \pi)$ 2.3 ± 0.1 $\mu(\omega \to \eta)$ 0.37 ± 0.15 $\mu(\eta' \to \omega)$ 0.44 ± 0.12 $\mu(\phi \to \eta)$ 0.69 ± 0.07 $\mu(\phi \to \eta')$ not known $\mu(K^{*0} \to K^0)$ 0.95 ± 0.22 $\mu(K^{*+} \to K^+)$ 0.86 ± 0.11

Normalize your calculation to $\mu(\omega \to \pi)$. Use $\eta = \frac{1}{2}(u\bar{u} + d\bar{d} - \sqrt{2}s\bar{s})$ and $\eta = \frac{1}{2}(u\bar{u} + d\bar{d} + \sqrt{2}s\bar{s})$.

2. The partial width for the Vector meson decay $\Gamma(V \to l^+l^-)$ is given by the Van Royen - Weisskopf formula:

$$\Gamma(V \to l^+l^-) = \frac{16\pi\alpha^2 Q^2}{M_V^2}|\psi(0)|^2$$

where $Q^2 = |\sum a_i Q_i|^2$ is the squared sum of the charges of the quarks in the meson, $\psi(0)$ is the wavefunction at the origin and M_V is the meson mass. Assumming that for the ρ, ω, and ϕ $|\psi(0)|^2/M_V^2$ is more or less constant, calculate the ratio of the ρ^0, ω and ϕ leptonic widths. Compare to experiment.

Chapter 5

Baryon Spectroscopy

Given the success of the quark model in the meson sector we turn to the next simplist system, the baryons. Baryons are better understood experimentally than mesons with various complete multiplets and many well investigated decay channels. This is because they are produced in S-channel πN and $\bar{K}N$ scattering while S-channel meson production is only possible for $J^{PC} = 1^{--}$ resonances in e^+e^-.

Baryons are more complicated than mesons because there are now 3 flavours and 3 spins and two relative coordinates which must be combined with colour to give a totally antisymmetric baryon wavefunction, resulting in much more complicated combinations of the various degrees of freedom. The baryon wavefunction is a product of

$$\begin{aligned} \psi &= (space) \times (spin) \times (flavour) \times (colour) \\ &= \psi \times \chi \times \phi \times C \end{aligned} \tag{5.1}$$

where $C_{ijk} = \frac{1}{\sqrt{6}}\epsilon_{ijk}|q_i q_j q_k\rangle$.

Let us start with the spatial part of the wavefunction. In what follows we will argue that we can approximate the forces between quarks in a baryon with two-body forces. To see that this is reasonable in some approximation, consider a quark pair with large separation from the 3rd remaining quark, fig. The resulting colour flux is essentially that of a meson since, to form a colour singlet, the diquark must combine to a colour $\bar{3}$ which combines with the colour 3 of the remaining quark.

Figure 5.1:

Thus, we replace the diquark-quark potential with 2-body potentials having half the strength of the $q\bar{q}$ potential in a meson:

$$v_{qq} = \frac{1}{2} V_{q\bar{q}} \tag{5.2}$$

This is the same factor of 1/2 that applies to the one-gluon-exchange part of the potential.

Numerical studies also find that replacing the *Mercedes Benz* diagram with a sum of 2-body potentials to be reasonable.

Rather than solve the linear plus Coulomb potential we can follow the very successful approach of Isgur and Karl which displays all the important features of a more careful calculation and is also quantitatively successful.

Isgur and Karl used harmonic oscillator potentials which have the virtue that one can separate variables to turn the problem into a sum of two harmonic oscillators.

$$\begin{aligned} H &= \sum_{i=1}^{3} \left(m_i + \frac{p_i^2}{2m_i} \right) + \sum_{i<j} V_{conf}^{ij} + \sum_{i<j} H_{hyp}^{ij} + \sum_{i<j} X^{ij} \\ &= H_{si} + H_{sd} \end{aligned} \tag{5.3}$$

where

$$V_{conf}^{ij} = \frac{1}{2} K r_{ij}^2 + U(r_{ij}) \tag{5.4}$$

and $U(r_{ij})$ contains deviations from the harmonic oscillator potentials and is treated in perturbation theory. For example,

$$U(r_{ij}) = -\frac{2\alpha}{3r} + br - \frac{1}{2}kr^2 \tag{5.5}$$

The hyperfine interactions are given by:

$$H^{ij}_{hyp} = \frac{2}{3}\frac{\alpha_s}{m_i m_j}\left\{\frac{8\pi}{3}\vec{S}_i \cdot \vec{S}_j \delta^3(r_{ij}) + \frac{1}{r_{ij}^3}\left[\frac{3\vec{S}_i \cdot \vec{r}_{ij}\ \vec{S}_j \cdot \vec{r}_{ij}}{r_{ij}^2} - \vec{S}_i\vec{S}_j\right]\right\} \tag{5.6}$$

and X^{ij} represents all the neglected effects like $\vec{L} \cdot \vec{S}$ etc.

The baryon wavefucntion is of the form

$$\begin{aligned} |qqq\rangle &= (space) \times (spin) \times (flavour) \times (colour) \\ &= \psi \times \chi \times \phi \times C \end{aligned} \tag{5.7}$$

where $C_{ijk} = \frac{1}{\sqrt{6}}\epsilon_{ijk}|q_i q_j q_k\rangle$. We will use the constituent quark masses $m_u \simeq m_d \simeq 0.35$ GeV, $m_s \simeq 0.58$ GeV, and $m_c \simeq 1.5$ GeV. You should note that the quark masses are parameters of the model and not fundamental parameters of QCD.

5.1 The Spin-Independent Spectrum

The Hamiltonian for the spin-independent spectrum is given by

$$\begin{aligned} H &= \sum_{i=1}^{3}\left(m_i + \frac{p_i^2}{2m_i}\right) + \sum_{i<j}\left(\frac{1}{2}kr_{ij}^2 + U(r_{ij})\right) \\ &= H_0 + \sum i < jU(r_{ij}) \end{aligned} \tag{5.8}$$

We can solve H_0 exactly for the eigenstates of H_0 and then choosing k to minimize the perturbation U where U is treated by perturbation theory.

Our first step is to find the eigenstates of H_0. It is convenient to distinguish 2 cases; where all 3 quarks have equal masses (S=0 and

S=-3 sectors, *uuu*, *uud*, *udd*, *ddd*, and *sss*) and when one quark has a mass different from the other two.

Let us take $m_1 = m_2 = m$ and $m_3 = m'$. Referring to fig. we can obtain new coordinates:

$$\begin{aligned} \vec{R}_{CM} &= \frac{m(\vec{r}_1 + \vec{r}_2) + m'\vec{r}_3}{2m + m'} \\ \vec{\rho} &= \frac{1}{\sqrt{2}}(\vec{r}_1 - \vec{r}_2) \\ \vec{\lambda} &= \frac{1}{\sqrt{6}}(\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3) \end{aligned} \tag{5.9}$$

using the change of variables:

$$\begin{aligned} \vec{r}_1 &= \vec{R}_{CM} + \frac{m'}{2m + m'}\frac{\sqrt{6}}{2}\vec{\lambda} + \frac{1}{\sqrt{2}}\vec{\rho} \\ \vec{r}_2 &= \vec{R}_{CM} + \frac{m'}{2m + m'}\frac{\sqrt{6}}{2}\vec{\lambda}\frac{1}{\sqrt{2}}\vec{\rho} \\ \vec{r}_2 &= \vec{R}_{CM} - \frac{2m'}{2m + m'}\frac{\sqrt{6}}{2}\vec{\lambda} \end{aligned} \tag{5.10}$$

The change of variables gives the Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}M\dot{R}_{CM}^2 + \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}\left(\frac{3mm'}{2m + m'}\right)\dot{\lambda}^2 + \frac{3}{2}k\rho^2 + \frac{3}{2}k\lambda^2 \\ &= \frac{p_{CM}^2}{2M} + \frac{p_\rho^2}{2m_\rho} + \frac{3}{2}k\rho^2\frac{p_\lambda^2}{2m_\lambda} + \frac{3}{2}k\lambda^2 \\ &= H_{CM} + H_\rho + H_\lambda \end{aligned} \tag{5.11}$$

where

$$m_\rho = m \qquad m_\lambda = \frac{3mm'}{2m + m'} \qquad p_i = m_i\dot{r}_i \tag{5.12}$$

and $\begin{matrix} \omega_\rho = \sqrt{\frac{2k}{m_\rho}} & \omega_\lambda = \sqrt{\frac{2k}{m_\lambda}} \\ \alpha_\rho^2 = \sqrt{2km_\rho} & \alpha_\lambda^2 = \sqrt{3km_\lambda} \end{matrix}$ so

$$E_{n_\rho l_\rho} = (2n_\rho + l_rho + 3/2)\hbar\omega_rho \tag{5.13}$$

As stated above we can divide the resulting spectra into two cases

Figure 5.2:

- All 3 quarks have equal masses S=0 and S=-3 sectors; *uuu*, *uud*, *ddu*, *ddd*, and *sss*.
- Two equal and one unequal S=-1 and -2 sectors *uus*, *uds*, *dds*, *ssu*, *ssd*.

For case I we have

$$
\begin{aligned}
M &= 3m \\
\alpha &= (3km)^{1/4} \\
\rho_{\pm} &= \rho_1 \pm i\rho_2 \\
\lambda_{\pm} &= \lambda_1 \pm i\lambda_2 \\
\omega &= (3k/m)^{1/2} \qquad (5.14)
\end{aligned}
$$

The wavefunctions are given by

$$\Psi = \psi \frac{\alpha^3}{\pi^{3/2}} e^{-1/2\alpha^2(\rho^2+\rho^2)}(5.15)$$

where the ψ are given by

$$
\begin{array}{ll}
N=0 & \psi 00 = 1 \\
N=1 & \psi^{\rho}_{11} = \alpha\rho_+ \\
 & \psi^{\lambda}_{11} = \alpha_+ \\
N=2 & \psi_{00} = \frac{1}{\sqrt{3}}\alpha^2(\rho^2+\lambda^2-3\alpha^{-2}) \\
 & \psi^{\rho}_{00} = \frac{2}{\sqrt{3}}\alpha^2\vec{\rho}\cdot\vec{\lambda}
\end{array}
$$

The subscripts denote the spherical symmetry of the wavefunction. In general each of the oscillators can be excited independently. For example

Figure 5.3:

in the N=1 levels either the ρ or λ oscillator is excited. For higher levels we have to combine the excitations of the two oscillators to give us the total energy. For example, for N=2, we can either put the ρ into the N=2, L=0 level or put the λ into the N=2, L=0 level, or we could put both the ρ and λ into N=1, L=1 levels, or we could put either the ρ or λ into N=0, L=2 levels. For higher levels the number of possible states proliferates rapidly. The difficulty comes in combining the spatial, spin, and flavour wavefunctions to properly symmetrize the total baryon wavefunction.

The gross features of the spectrum are given in fig.

For case II we have $m_1 = m_2 = m_d < m_3 = m_s$ or $m_1 = m_2 = m_s > m_3 = m_d$. For $S = -1$

$$\begin{aligned}
m_\rho &= m \\
m_\lambda &= \frac{3m_1 m_3}{2m_1 + m_3} \\
\omega_\rho &= (3k/m)^{1/2} \\
\omega_\lambda &= \left(\frac{3k}{m}\frac{2m_1 + m_3}{3m_3}\right)^{1/2} \\
&= \left(\frac{3k}{m}\right)^{1/2}\sqrt{\frac{1}{3}\left(1 + \frac{2m_1}{m_3}\right)}
\end{aligned}$$

$$= \omega_\rho \sqrt{\frac{1}{3}\left(1 + \frac{2m_1}{m_3}\right)} \tag{5.16}$$

For $S = -1$ we have $\omega_\lambda < \omega_\rho$ and for $S = -2$ we have $\omega_\rho < \omega_\lambda$. This breaks much of the degeneracy. The wavefunctions are similar to the equal mass case.

5.2 Spin Dependent Effects

5.3 Baryon Couplings

5.3.1 Strong Couplings

5.3.2 EM Couplings

Problems

1. For the Baryon Hamiltonian with harmonic confinement forces sketch the spectrum for the case $m/m' = 0.6$ (the ratio of the masses of non-strange to strange quarks) side-by-side with the case $m = m'$.

2. Calculate the magnetic moments of the low lying baryons p, n, Λ, Σ^+, Σ^0, Σ^-, Ξ^0, Ξ^-, and Δ^{++} and the two transition moments $\langle p|\mu|\Delta^+\rangle$ and $\langle \Lambda|\mu|\Sigma^0\rangle$ and compare to experiment in a table. Most experimental values are in the particle data book. The missing ones which are known are (in nuclear magnetons) $\mu_{\Delta^{++}} = +5.7 \pm 1.0$, $\mu_{N\Delta} = 3.76 \pm 0.19$, and $\mu_{\Lambda\Sigma} = 1.61 \pm 0.08$.

3. Using the expression

$$M(baryon) = m_1 + m_2 + m_3 + A' \left[\frac{\vec{S}_1 \cdot \vec{S}_2}{m_1 m_2} + \frac{\vec{S}_1 \cdot \vec{S}_3}{m_1 m_3} + \frac{\vec{S}_2 \cdot \vec{S}_3}{m_2 m_3}\right] \tag{5.17}$$

Derive expressions for the masses of the ground state baryons: Λ, Σ, Σ^*, N, P, Δ, Ω Ξ and Ξ^*. Verify the equal spacing rule:

$$\Sigma^* - \Delta = \Xi^* - \Sigma^* = \Omega - \Xi^* \tag{5.18}$$

and the Gell-Mann-Okubo relation

$$\frac{N + \Xi}{2} = \frac{3\Lambda + \Sigma}{4} \tag{5.19}$$

Chapter 6

Multiquark States

6.1 $qq\bar{q}\bar{q}$ Molecules

6.2 Final State Interactions

6.3 6 Quarks: Nuclear Physics from the Quark Model

Chapter 7

Heavy Quark Effective Theory (not included)

Chapter 8

Hadron Spectroscopy Beyond the Quark Model

8.1 QCD revisted

8.2 Quantum Numbers

8.3 Models of Glueballs and Hybrids

8.4 Gluonium

8.5 Hybrids

8.5.1 Hybrid Decays

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